Propositional Logic (Review)

CS 579

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1 Syntax

Formulas (or *sentences*) are constructed from *propositions* (or *statements*) and the logical connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

The BNF grammar of formulas in propositional logic is given by:

```
Formula         →   Proposition | ComplexFormula
Proposition     →   True | False | Symbol
Symbol          →   p | q | r | ...
ComplexFormula  →   ¬ Formula |
                    (Formula ∧ Formula) |
                    (Formula ∨ Formula) |
                    (Formula → Formula) |
                    (Formula ↔ Formula)
```

(Sometime, the parentheses can be omitted if no confusion is possible.)

2 Semantics

Interpretation. An interpretation I maps each symbol p (proposition symbol) into a value $I(p) \in \{True, False\}$. (sometime, we write t and f instead of $True$ and $False$.)

p is true in an interpretation I if $I(p) = True$.

The truth value of a formula φ in an interpretation I is determined by the truth value of the proposition symbols occurring in φ and the truth table:

p	q	¬ p	p ∧ q	p ∨ q	p → q	p ↔ q
t	t	f	t	t	t	t
t	f	f	f	t	f	f
f	t	t	f	t	f	f
f	f	t	f	t	f	t

Satisfiable. A formula φ is *satisfiable* if there exists an interpretation I and φ is true in I. (I will be called as a *model* of φ).

Validity. A formula φ is *valid* if φ is true in every interpretation (of φ).

Knolwedge Base. A KB is a set of formulas.

Model. An interpretation I is a *model* of KB if every formula belonging to KB is true in I.

Entailment. A KB *entails* a formula φ, denoted by $KB \models \varphi$, if φ is true in every model of KB.

Equivalence. Two formulas ϕ and ψ are equivalent if and only if $\phi \leftrightarrow \psi$ is a valid formula.

3 Inference

Given a KB and a formula φ, to determine whether $KB \models \varphi$ holds or not, we can use one of the following methods.

- *Model theoretic approach:* We can find all the models of KB and check whether φ is true in them; if φ is false in some model of KB, then the answer is *no*; otherwise, the answer is *yes*.
- *Proof theoretic approach:* We can also use *inference rules* such as
  1. The *modus ponents* rule:
  $$\frac{\phi \rightarrow \psi, \phi}{\psi}$$
  which says that whenever $\phi \rightarrow \psi$ and ψ are true, then so is ψ;
  2. The *and elimination* rule:
  $$\frac{\phi \wedge \psi}{\phi};$$
  3. The *resolution* rule:
  $$\frac{p_1 \vee \ldots \vee p_k, m}{p_1 \vee \ldots \vee p_{i-1} \vee p_{i+1} \vee \ldots \vee p_k}$$
  where each p_j is literal (a proposition p or its negation $\neg p$) and m is the complementary literal of p_i (i.e., if $p_i = p$ (resp. $p_i = \neg p$) then $m = \neg p$ (resp. $m = p$)).

  φ is a *consequence* of KB with respect to a set of inference rules S if we can use the inference rules in S to obtain φ from KB. We will write $KB \vdash_S \varphi$ to say that φ is derived from KB wrt. S.

A set of inference rules S is *sound* if $KB \vdash_S \varphi$ implies that $KB \models \varphi$. It is *complete* if $KB \models \varphi$ then $KB \vdash_S \varphi$ holds.

For propositional logic, the resolution rule is complete.