CS579 - Homework 2

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March 10, 2004

1 Review

1.1 Planning Problem and Plans

The planning problem P we are considering is a 3-tuple $\langle D, I, G\rangle$ described in the $\mathcal{ADL}$ language whose syntax is given in the book, where I is the initial state, G is the goal and D is the domain description.
Plans. We define a plan π as a sequence of sets of actions, i.e., $\pi = A_0; A_1; ...; A_{n-1}$ where A_i, $0 \le i \le n-1$, is a set of actions. That is, we allow actions to be executed parallelly.
Transition Function. Given a planning problem $P = \langle D, I, G\rangle$ with the set of actions $\mathbf{A}$ and set of fluents $\mathbf{F}$, the transition function Φ of the planning domain that maps sets of actions and states into states is defined as follows.

$$\Phi(A,\sigma) = \begin{cases} \perp & \text{if } \sigma = \perp \text{ or } \exists a \in A \text{ s.t. } a \text{ is not executable in } \sigma \text{ or } Add_A \cap Del_A \neq \emptyset \\ (\sigma \cup Add_A) \setminus Del_A & \text{otherwise} \end{cases} \tag{1}$$

where $A \subseteq \mathbf{A}$ is some set of actions, s is a state, $Add_A = \bigcup_{a\in A}(Add_a)$, and $Del_A = \bigcup_{a\in A}(Del_a)$.

It is worthwhile extending the definition of Φ for a plan π as well. More precisely, the transition function Φ for plans are defined recursively as follows:

$$\Phi([A],\sigma) = \Phi(A,\sigma)$$
$$\Phi([A;p],\sigma) = \Phi(p, \Phi(A,\sigma)) \text{ if } p \neq \emptyset$$

$\sigma_0 = I$ is called the initial state. We say that G can be *achieved* from σ_0 after executing plan π if $\Phi(\pi,\sigma_0) \neq \perp$ and $\Phi(\pi,\sigma_0)$ satisfies G.

1.2 SAT Encodings

Given a planning problem $P = \langle D, I, G\rangle$ in the $\mathcal{ADL}$ language with the transition function given in (1), the general procedure for translating P into the corresponding propositional theory T with n time instants is sketched out as follows:

- Each fluent $f \in \mathbf{F}$ corresponds to the set of propositions $\{f^i\}$ in T, where $i = 0 \ldots n$.
- Each action $a \in \mathbf{A}$ corresponds to the set of propositions $\{a^i\}$, where $i = 0 \ldots n-1$.
- The encoding consists of the following sentences

$$\bigwedge_{f\in I} f^0 \tag{2}$$

$$\bigwedge_{g\in G} g^n \tag{3}$$

$$a^i \Rightarrow \bigwedge_{l\in Pre_a} l^i \qquad \forall a \in \mathbf{A}, i = 0\ldots n-1 \tag{4}$$

$$f^{i+1} \leftrightarrow (f^i \wedge (\bigwedge_{a\in\mathbf{A}, f\in Del_a} \neg a^i)) \vee (\bigvee_{a\in\mathbf{A}, f\in Add_a} a^i) \qquad \forall f \in \mathbf{F}, i = 0\ldots n-1 \tag{5}$$

$$\neg f^{i+1} \leftrightarrow (\neg f^i \wedge (\bigwedge_{a\in\mathbf{A}, f\in Add_a} \neg a^i)) \vee (\bigvee_{a\in\mathbf{A}, f\in Del_a} a^i) \qquad \forall f \in \mathbf{F}, i = 0\ldots n-1 \tag{6}$$

1.3 Correctness

We will show that the encoding T from (2)-(6) is equivalent to the original planning problem $P = \langle D, I, G \rangle$.

2 Proof of the Correctness

To prove the correctness of the SAT-encoding, we need prove the following

1. [Soundness] If T has a model M then we can extract from M a plan π for P that achieves G from the initial state I.
2. [Completeness] If P has a plan π of length n that achieves G from the initial state then T has at least one model M from which π can be extracted.

2.1 Soundness

Assume that M is a model of T. Let $A_i = \{a \mid a \text{ is an action such that } a^i \in M\}$, $\pi = [A_0; A_1; \ldots A_{n-1}]$ and $s_i = \{f \mid f^i \in M\}$. Note that $A_i's$ might be empty. By (2) and (3), it is easy to see that s_0 is the initial state of P and G holds in s_n. Before proving the soundness of the encoding, let us prove the following lemmas.

Lemma 1 *For every integer i, $0 \le i \le n$, a fluent literal l holds in a state s_i if and only if $M \models l^i$.*

Proof. It is obvious from the definition of s_i.

Lemma 2 *For every integer i, $0 \le i \le n-1$, and for every action a in A_i, a is executable in s_i.*

Proof. Let l be a fluent literal in Pre_a. By the definition of A_i, we have $M \models a^i$. On the other hand, $a^i \Rightarrow l^i$ (by (4)). Hence, $M \models l^i$. As a result, l^i holds in s_i (by Lemma 1). That is, a is executable in s_i.

Lemma 3 *For every integer i, $0 \le i \le n-1$, $Add_{A_i} \cap Del_{A_i} = \emptyset$.*

Proof. Assume that $Add_{A_i} \cap Del_{A_i} \neq \emptyset$. Thus, there exists a fluent $f \in \mathbf{F}$ s.t. $f \in Add_{A_i} \cap Del_{A_i}$. This implies that there exist two actions $b, c \in A_i$ s.t. $f \in Add_b$ and $f \in Del_c$.

By (5), we have f^{i+1} is true in M. And by (6), we have $\neg f^{i+1}$ is true in M. This is a contradiction. Therefore, $Add_{A_i} \cap Del_{A_i} = \emptyset$.

Lemma 4 *For every integer i, $0 \le i \le n-1$, we have $\Phi(A_i, s_i) = s_{i+1}$.*

Proof. According to Lemma 2 and Lemma 3, every action a in A_i is executable in s_i and $Add_{A_i} \cap Del_{A_i} = \emptyset$ for all $0 \le i \le n-1$. Therefore, according to (1), $\Phi(A_i, s_i) = (s_i \cup Add_{A_i}) \setminus Del_{A_i} = s$ for some s. We will prove $s_{i+1} = s$ by showing that, for every $f \in \mathbf{F}$,

(1) if $f \in s_{i+1}$ then $f \in s$

(2) if $f \notin s_{i+1}$ then $f \notin s$

Proof of (1). Assume that $f \in s_{i+1}$.

According to Lemma 1, we have $M \models f^{i+1}$. Hence,

$$M \models (f^i \wedge (\bigwedge_{a \in \mathbf{A}, f \in Del_a} \neg a^i)) \vee (\bigvee_{a \in \mathbf{A}, f \in Add_a} a^i)$$

There are two possibilities:

a) $M \models (f^i \wedge (\bigwedge_{a \in \mathbf{A}, f \in Del_a} \neg a^i))$
It is easy to see that $M \models f^i$. Thus, $f \in s_i$.

b) $M \models \bigvee_{a \in \mathbf{A}, f \in Add_a} a^i$. That is, there must exist an action b such that $M \models b^i$ and $f \in Add_b$. Since $M \models b^i$, we have $b \in A_i$. Hence, $f \in Add_b \subseteq Add_{A_i}$. In addition, $Add_{A_i} \cap Del_{A_i} = \emptyset$ (Lemma 4), we have $f \notin Del_{A_i}$. As a result $f \in (s_i \cup Add_{A_i}) \setminus Del_{A_i} = s$.

Therefore, in both cases, we have $f \in s$.

Proof of (2). Assume that $f \notin s_{i+1}$.

According to Lemma 1, f^{i+1} is not in M, i.e., $M \models \neg f^{i+1}$. By (6), we have two possibilities:

a) $M \models (\neg f^i \wedge (\bigwedge_{a \in \mathbf{A}, f \in Add_a} \neg a^i))$. In this case, we have $M \models \neg f^i$. Therefore, $f \notin s_i$ (by Lemma 1). On the other hand, since $M \models (\neg f^i \wedge (\bigwedge_{a \in \mathbf{A}, f \in Add_a} \neg a^i))$, we have $M \models (\bigwedge_{a \in \mathbf{A}, f \in Add_a} \neg a^i))$. Thus, $f \notin Add_{A_i}$. Consequently, $f \notin (s_i \cup Add_{A_i}) \setminus Del_{A_i} = s$.

b) $M \models (\bigvee_{a \in \mathbf{A}, f \in Del_a} a^i)$. That is, there must be exist an action b such that $f \in Del_b$ and b^i is true in M. That is, $f \in Del_{A_i}$ and thus $f \notin (s_i \cup Add_{A_i}) \setminus Del_{A_i} = s$

So, we can conclude that if $f \notin s_{i+1}$ then $f \notin s$.

The lemma is therefore proved.

Theorem 1 *The encoding is sound.*

Proof. Let $\sigma_i = \Phi(A_0 \ldots A_{i-1}, \sigma_0)$ for $1 \leq i \leq n$. As we mentioned above, $\sigma_0 = s_0$. By Lemma 4, using induction, we can easily prove that $s_i = \sigma_i$ for all $0 \leq i \leq n$. Hence, $\Phi(\pi, \sigma_0) = \sigma_n = s_n$. On the other hand, G holds s_n. Thus, G holds in $\Phi(\pi, \sigma_0)$. That is, G can be achieved from σ_0 after executing π.

The encoding is therefore proved to be sound.

2.2 Completeness

Assume that P has a plan $\pi = A_0; A_1; ...; A_{n-1}$, where A_i, $0 \leq i \leq n-1$, is a set of actions, that achieves goal G from σ_0. We denote $\Phi(A_0; A_1; ...; A_{j-1}, \sigma_0)$ by σ_i. Since π is a plan that achieves goal G, we have the following properties:

(1) $\sigma_0 = I$.

(2) for every integer, i, $0 \leq i \leq n-1$, and for every action $a \in A_i$, a is executable in σ_i.

(3) $\sigma_i \neq \perp$ and $Del_{A_i} \cap Add_{A_i} = \emptyset$ for every i, $0 \leq i \leq n$

(4) $\sigma_{i+1} = (\sigma_i \cup Add_{A_i}) \setminus Del_{A_i}$.

(5) for every literal $l \in G$, l holds in σ_n.

Let T be the corresponding propositional theory for P with n time instants and for every i, $0 \leq i \leq n$, let δ_i denote the set of propositions $\{f^i \mid f \in \sigma_i\}$. We construct an interpretation M of T as follows:

$$M = (\bigcup_{0 \leq i \leq n} \delta_i) \cup (\bigcup_{0 \leq i \leq n-1} \{a^i | a \in A_i\})$$

First of all, we will prove the following lemmas and then use them to prove the completeness of the encoding.

Lemma 5 *For every integer i, $0 \leq i \leq n$, a fluent literal l holds in σ_i iff $M \models l^i$.*

Proof. It is easy to see that $M \models l^i$ iff l^i is contained in δ_i, that is, l holds in σ_i.

Lemma 6 *For every integer i, $0 \leq i \leq n-1$, we have*

$$M \models a^i \qquad \forall a \in A_i$$

$$M \models \neg a^i \qquad \forall a \notin A_i$$

Proof. It is obvious from the construction of M.

Theorem 2 *The encoding is complete.*

Proof. To prove that the encoding is complete we need to prove

(1) M is a model of T and,

(2) from M we can construct the plan π.

Proving (2) is trivial. We will show (1) by checking all the propositional sentences in T.

By Lemma 5, the sentence (2) and (3) are apparently true in M.

For every integer i, $0 \leq i \leq n-1$, and an action $a \in A_i$, since a is executable in σ_i, every fluent literal $l \in Pre_a$ must hold in σ_i. By Lemma 5, we have $M \models l^i$. Thus, the sentence (4) is true in M.

¿From Lemma 6, we have:

$$\bigwedge_{a \in \mathbf{A}, f \in Del_a} \neg a^i \leftrightarrow \bigwedge_{a \in A_i, f \in Del_a} \neg a^i$$

$$\bigwedge_{a \in \mathbf{A}, f \in Add_a} \neg a^i \leftrightarrow \bigwedge_{a \in A_i, f \in Add_a} \neg a^i$$

$$\bigvee_{a \in \mathbf{A}, f \in Add_a} a^i \leftrightarrow \bigvee_{a \in A_i, f \in Add_a} a^i$$

$$\bigvee_{a \in \mathbf{A}, f \in Del_a} a^i \leftrightarrow \bigvee_{a \in A_i, f \in Del_a} a^i$$

So, to prove the sentences (5) and (6) are true in M, we only need to prove the following sentences are true in M.

$$f^{i+1} \leftrightarrow (f^i \wedge (\bigwedge_{a \in A_i, f \in Del_a} \neg a^i)) \vee (\bigvee_{a \in A_i, f \in Add_a} a^i) \tag{7}$$

$$\neg f^{i+1} \leftrightarrow (\neg f^i \wedge (\bigwedge_{a \in A_i, f \in Add_a} \neg a^i)) \vee (\bigvee_{a \in A_i, f \in Del_a} a^i) \tag{8}$$

For each fluent $f \in \mathbf{F}$, and an integer i, $0 \leq i \leq n-1$, consider the following four cases:

Case 1. *f holds in both σ_i and σ_{i+1}.*

According to Lemma 5, we have $M \models f^i$ and $M \models f^{i+1}$.

Since $\sigma_{i+1} = (\sigma_i \cup Add_{A_i}) \setminus Del_{A_i}$, $f \notin Del_{A_i}$. Hence, we can reduce (7) and (8) to:

$$f^{i+1} \leftrightarrow f^i \vee (\bigvee_{a \in A_i, f \in Add_a} a^i) \tag{9}$$

$$\neg f^{i+1} \leftrightarrow (\neg f^i \wedge (\bigwedge_{a \in A_i, f \in Add_a} \neg a^i)) \tag{10}$$

(9) is true in M as both f^i and f^{i+1} are true in M. (9) is also true in M as both the left hand side and the right hand side are false in M.

Case 2. *f holds in σ_i but does not hold in σ_{i+1}.*

By Lemma 5, we have $M \models f^i$ and $M \not\models f^{i+1}$.

Since $\sigma_{i+1} = (\sigma_i \cup Add_{A_i}) \setminus Del_{A_i}$, $f \in Del_{A_i}$. So, there exists $a \in A_i$ such that $f \in Del_a$. Thus, $(\bigvee_{a \in A_i, f \in Del_a} \neg a^i)$ is false in M and $(\bigwedge_{a \in A_i, f \in Del_a} a^i)$ are true in M. On the other hand, since $Del_{A_i} \cap Add_{A_i} = \emptyset$, we have $f \notin Add_{A_i}$. As a result, both sides of (7) are false, whereas both sides of (8) are true. That is, (7) and (8) are true in M.

Case 3. *f does not hold in σ_i but holds in σ_{i+1}.*

In this case, we have f^{i+1} is true in M but f^i is not. Since, $\sigma_{i+1} = (\sigma_i \cup Add_{A_i}) \setminus Del_{A_i}$, we have $f \in Add_{A_i}$. Thus $f \notin Del_{A_i}$. Similarly to Case 2, (7) is true in M as its both sides are true and (8) is also true because its both sides are false.

Case 4. *f does not hold in both σ_i and σ_{i+1}.*

In this case, we have $M \models \neg f^i$ and $M \models \neg f^{i+1}$. Since $\sigma_{i+1} = (\sigma_i \cup Add_{A_i}) \setminus Del_{A_i}$, we have either $f \in Del_{A_i}$ or $f \notin Add_{A_i}$. However, $f \in Del_{A_i}$ also implies that $f \notin Add_{A_i}$. Hence, in both cases, we have $f \notin Add_{A_i}$.

(7) and (8) are therefore equivalent to

$$f^{i+1} \leftrightarrow f^i \wedge (\bigwedge_{a \in A_i, f \in Del_a} \neg a^i) \tag{11}$$

$$\neg f^{i+1} \leftrightarrow \neg f^i \vee (\bigvee_{a \in A_i, f \in Del_a} a^i) \tag{12}$$

It is easy to see that both of them are true in M.
M is therefore a model of T. So, the encoding is complete.