Exercise 5.15

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We can assume that $x_u^2 + y_u^2 + z_u^2 == 1$ since rotating θ degree about the direction specified by U is the same as rotating θ degree about the direction specified by the unit vector along U. The following steps are involved:

- **Step 1:** Rotate about y-axis so U lies on the yz-plan. This is done as on page 218 of the book (so the point P_2' lies on the yz-plan). The rotation matrix for this step is:

$$R_y(\alpha - 90°) = \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{-x_u}{D_1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_u}{D_1} & 0 & \frac{z_u}{D_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  where α is the angle between the projection of U and the x-axis and D_1 is the distance from the projection of U onto the xz-plane to the origine, which is $\sqrt{x_u^2 + z_u^2}$.

  Notice that the point U will go to U' whose coordinates are given by

$$\begin{bmatrix} 0 \\ y_u \\ D_1 \\ 1 \end{bmatrix}$$

  because $U' = R_x(\alpha - 90°).U$.

  (this is the point P_2'' on page 218)

- **Step 2:** Rotate about x-axis so U lies on the z-axis. This is done as on page 219 of the book (so the point P_2'' lies on the z-axis). The rotation matrix for this step is:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D_1 & -y_u & 0 \\ 0 & y_u & D_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  (this is because the distance from U' to the origine, is $\sqrt{x_u^2 + y_u^2 + z_u^2}$ and equals 1.

- **Step 3:** Rotate θ about the z-axis, this gives the matrix

$$R_z(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Step 4:** The inverse of $R_x(\phi)$ is $R_x(-\phi)$ which is

$$R_x(-\phi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D_1 & y_u & 0 \\ 0 & -y_u & D_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Step 5:** The inverse of $R_y(\alpha - 90^\circ)$ is $R_y(90^\circ - \alpha)$ which is

$$R_y(90^\circ - \alpha) = \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{x_u}{D_1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-x_u}{D_1} & 0 & \frac{z_u}{D_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the composite matrix is

$$R_y(90^\circ - \alpha).R_x(-\phi).R_z(\theta).R_x(\phi).R_y(\alpha - 90^\circ)$$

The step of computation is given next.

- Computing M:

$$\begin{aligned} M = R_x(\phi).R_y(\alpha - 90^\circ) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D_1 & -y_u & 0 \\ 0 & y_u & D_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{-x_u}{D_1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_u}{D_1} & 0 & \frac{z_u}{D_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{-x_u}{D_1} & 0 \\ \frac{-x_u y_u}{D_1} & D_1 & \frac{-y_u z_u}{D_1} & 0 \\ x_u & y_u & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

- Computing $T = R_z(\theta).M$

$$T = R_z(\theta).M = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{-x_u}{D_1} & 0 \\ \frac{-x_u y_u}{D_1} & D_1 & \frac{-y_u z_u}{D_1} & 0 \\ x_u & y_u & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{z_u cos\theta}{D_1} + \frac{x_u y_u sin\theta}{D_1} & -D_1 sin\theta & \frac{-x_u cos\theta}{D_1} + \frac{y_u z_u sin\theta}{D_1} & 0 \\ \frac{z_u sin\theta}{D_1} - \frac{x_u y_u cos\theta}{D_1} & cos\theta D_1 & \frac{-x_u sin\theta}{D_1} + \frac{-y_u z_u cos\theta}{D_1} & 0 \\ x_u & y_u & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Computing M^{-1}:

$$\begin{aligned} M^{-1} = R_y(90^\circ - \alpha).R_x(-\phi) &= \begin{bmatrix} \frac{z_u}{D_1} & 0 & \frac{x_u}{D_1} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-x_u}{D_1} & 0 & \frac{z_u}{D_1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D_1 & y_u & 0 \\ 0 & -y_u & D_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{z_u}{D_1} & \frac{-x_u y_u}{D_1} & x_u & 0 \\ 0 & D_1 & y_u & 0 \\ \frac{-x_u}{D_1} & \frac{-y_u z_u}{D_1} & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

- The composite matrix: $C = M^{-1}R_z(\theta)M$

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\begin{aligned}
C = M^{-1}T &= \begin{bmatrix} \frac{z_u}{D_1} & \frac{-x_u y_u}{D_1} & x_u & 0 \\ 0 & D_1 & y_u & 0 \\ \frac{-x_u}{D_1} & \frac{-y_u z_u}{D_1} & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{z_u cos\theta}{D_1} + \frac{x_u y_u sin\theta}{D_1} & -D_1 sin\theta & \frac{-x_u cos\theta}{D_1} + \frac{y_u z_u sin\theta}{D_1} & 0 \\ \frac{z_u sin\theta}{D_1} - \frac{x_u y_u cos\theta}{D_1} & cos\theta D_1 & \frac{-x_u sin\theta}{D_1} + \frac{-y_u z_u cos\theta}{D_1} & 0 \\ x_u & y_u & z_u & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} r_{1x} & r_{2x} & r_{3x} & 0 \\ r_{1y} & r_{2y} & r_{3y} & 0 \\ r_{1z} & r_{2z} & r_{3z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
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We have

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\begin{aligned}
r_{1x} = {} & \frac{z_u}{D_1}\left(\frac{z_u cos\theta}{D_1} + \frac{x_u y_u sin\theta}{D_1}\right) + \frac{-x_u y_u}{D_1}\left(\frac{z_u sin\theta}{D_1} - \frac{x_u y_u cos\theta}{D_1}\right) + x_u^2 \\
= {} & \frac{z_u^2 cos\theta}{D_1^2} + \frac{x_u y_u z_u sin\theta}{D_1^2} + \frac{-x_u y_u x_u sin\theta}{D_1^2} + \frac{x_u^2 y_u^2 cos\theta}{D_1^2} + x_u^2 = \frac{z_u^2 cos\theta}{D_1^2} + \frac{x_u^2 y_u^2 cos\theta}{D_1^2} + x_u^2
\end{aligned}
$$

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\begin{aligned}
r_{1x} = {} & \frac{z_u^2 cos\theta}{D_1^2} + \frac{x_u^2 y_u^2 cos\theta}{D_1^2} + x_u^2 = x_u^2 + cos\theta\frac{z_u^2 + x_u^2 y_u^2}{D_1^2} = x_u^2 + cos\theta\frac{z_u^2 + x_u^2(1 - D_1^2)}{D_1^2} \\
= {} & x_u^2 + cos\theta\frac{z_u^2 + x_u^2 - x_u^2 D_1^2)}{D_1^2} = x_u^2 + cos\theta\frac{D_1^2 - x_u^2 D_1^2)}{D_1^2} = x_u^2 + cos\theta(1 - x_u^2)
\end{aligned}
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\begin{aligned}
r_{2x} = {} & \frac{z_u}{D_1} - D_1 sin\theta + \frac{-x_u y_u}{D_1} D_1 cos\theta + x_u y_u \\
= {} & -z_u sin\theta + -x_u y_u cos\theta + x_u y_u = x_u y_u(1 - cos\theta) - z_u sin\theta
\end{aligned}
$$

 and so $r_{2x} = x_u y_u(1 - cos\theta) - z_u sin\theta$ if $= 1$.

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\begin{aligned}
r_{3x} = {} & \frac{z_u}{D_1}\left(\frac{-x_u cos\theta}{D_1} + \frac{y_u z_u sin\theta}{D_1}\right) + \frac{-x_u y_u}{D_1}\left(\frac{-x_u sin\theta}{D_1} + \frac{-y_u z_u cos\theta}{D_1}\right) + x_u z_u \\
= {} & x_u z_u + \frac{y_u z_u^2 sin\theta + x_u z_u y_u^2 cos\theta + x_u^2 y_u sin\theta - x_u z_u cos\theta}{D_1^2} \\
= {} & z_u z_u + \frac{y_u(x_u^2 + z_u^2) sin\theta + x_u z_u(1 - y_u^2) cos\theta}{D_1^2} = x_u z_u(1 - cos\theta) + y_u sin\theta
\end{aligned}
$$