Representing Curves and Surfaces

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1 Introduction

Curves and surfaces are an integral part of almost everything that is displayed and designed on computer. We often need to approximate a curve or a surface due to

- a curve or a surface contains infinitely many points and computer display (no matter how big is it or what resolution is it) is finite;
- sometimes we do not know the exact function that describes the curve or the surface.

2 Polygon Meshes

A *polygon mesh* approximates a curve (or a surface) by a collection of edges, vertices, and polygons connected such that each edge is shared by **at most two** polygons.

There are different representations for a polygon mesh.

- **Explicit representation**: Each polygon is represented by a list of vertex coordinates:

$$P = ((x_1, y_1, z_1), \ldots, (x_n, y_n, z_n))$$

- **Pointers to a vertex list**: A polygon mesh is represented by
  - a list of vertices
  $$V = ((x_1, y_1, z_1), \ldots, (x_n, y_n, z_n))$$
  - each polygon is represented by a list of indices, e.g.,
  $$P = (0, 1, 3, 4)$$
  represents a polygon whose vertices are the first, second, fourth, and fifth in the list V.
- **Pointers to a edge list**: A polygon mesh is represented by
  - a list of vertices
  $$V = ((x_1, y_1, z_1), \ldots, (x_n, y_n, z_n))$$
  - each edge is represented by a list of two indices (two vertices) and the polygon containing the edge, e.g.,
  $$E = (0, 1, P_1, P2)$$
  the edge connecting the first and second vertice (in V) belong to the polygons P_1 and P_2.

There are two poblems that should be considered when a polygon mesh is used in representing a curve or a surface.

1. **Consistency:** the definition indicates an edge is shared by at most two polygons. If a representation does not satisfy this property, it is inconsistent.
2. **Plane equation:** The points representing a polygon might or might not lie on a plane. If the points (in the real world) are lying on a plane and their representation in a polygon mesh does not satisfy this property, we have an inconsistent representation as well. In this case, we need to be able to identify an approximation for the representation. A plane that has minimal total sum of distances of the points is often used as the approximation.

3 Parametric Cubic Curves

The smoothness of the curve or surface representing by polygon meshes depends on the number of polygons. Another approach to represent a curve is to use parametric cubic curve. This is given by the three functions

$$\begin{aligned} x(t) &= a_x t^3 + b_x t^2 + c_x t + d_x, \quad (1) \\ y(t) &= a_y t^3 + b_y t^2 + c_y t + d_y, \quad (2) \\ z(t) &= a_z t^3 + b_z t^2 + c_z t + d_z. \quad (3) \end{aligned}$$

for $0 \leq t \leq 1$.

We can rewrite the above in matrix form as

$$Q(t) = T.C \quad (4)$$

where

- $Q(t) = [x(t)\ y(t)\ z(t)]$,
- $T = [t^3\ t^2\ t\ 1]$, and
- C is the coefficient matrix,

$$C = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}$$

The above equation can be further rewritten into the following form

$$Q(t) = T.M.G \quad (5)$$

where

- $Q(t) = [x(t)\ y(t)\ z(t)]$,
- $T = [t^3\ t^2\ t\ 1]$,
- M is a 4×4 basic matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

- G is called the *geomertiy vector* and is a four-element column vector representing the constraints on the curve. It could be the endpoints, the tangent vectors, etc. In general,

$$G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix}$$

We use G_x to denote the column vector of the x components of the geometry vector G. This gives: $x(t) = T.M.G_x$. Expanding this gives:

$$\begin{aligned} x(t) \quad = \quad & (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} \\ & +(t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x} \\ & +(t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} \\ & +(t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x} \end{aligned}$$

which says that $x(t)$ is a weighted sum of the elements of the geometry matrix. The weights are each cubic polynomials of t and are called the *blending functions*, which will be denoted by B and equals $T.M$.

There are several special cases of the general form 5. A few of them are:

- **Hermite Curve:** The geometry vector G_H is given by two endpoints, P_1 and P_4, and two tangent vectors at the endpoints R_1 and R_4. The coefficent matrix M_H can be determined by a series of simple computation and is

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- **Bézier Curve:** The geometry vector G_B is given by four points, P_1, P_2, P_3, and P_4 with the constraints relating to the tangent vectors R_1 and R_4 by $R_1 = 3(P_2 - P_1)$ and $R_4 = 3(P_4 - P_3)$. The coefficent matrix M_B is

$$M_B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- **B-spline Curve:** approximating a curve given by multiple points $P_0, P_1, \ldots, P_m$ by multiple segments $Q_3, Q_4, \ldots, Q_m$. Can be uniform or non-uniform. For uniform case, the basic matrix M_{Bs} is

$$M_{Bs} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

  and the geometry vector for Q_i is

$$G_{Bs_i} = \begin{bmatrix} P_{i-3} \\ P_{i-2} \\ P_{i-1} \\ P_i \end{bmatrix}$$

There are codes for drawing a parametric cubic curves. Method for incremental drawing the curve is presented also.

4 Parametric Cubic Surfaces

Like with the curves, we represent a surface by a function

$$Q(s) = S.M.G$$

where $S = [s^3\ s^2\ s\ 1]$, M is a 4×4 coefficient matrix, and G is the geometry vector. If we allow the points in G to vary along a path that is parameters on t, we have

$$Q(s,t) = S.M. \begin{bmatrix} G_1(t) \\ G_2(t) \\ G_3(t) \\ G_4(t) \end{bmatrix}$$

If each $G_i(t)$ is a cubic, we say that $Q(s,t)$ is a *parametric cubic surface*.

Each $G_i(t)$ can be written as $G_i(t) = T.M.\mathbf{G_i}$ where $\mathbf{G_i} = [\mathbf{g_{i1}}\ \mathbf{g_{i2}}\ \mathbf{g_{i3}}\ \mathbf{g_{i4}}]^T$. Rewritten this for $Q(s,t)$, we get

$$Q(s,t) = S.M. \begin{bmatrix} \mathbf{g_{11}}\ \mathbf{g_{12}}\ \mathbf{g_{13}}\ \mathbf{g_{14}} \\ \mathbf{g_{21}}\ \mathbf{g_{22}}\ \mathbf{g_{23}}\ \mathbf{g_{24}} \\ \mathbf{g_{31}}\ \mathbf{g_{32}}\ \mathbf{g_{33}}\ \mathbf{g_{34}} \\ \mathbf{g_{41}}\ \mathbf{g_{42}}\ \mathbf{g_{43}}\ \mathbf{g_{44}} \end{bmatrix} .M^T.T^T$$

Like with surface, the above formula is the general formula for representing a surface. We can consider the different special cases such as the Hermite surface, the Bézier surface, or the B-spline surface.

There are also codes for drawing a parametric cubic surfaces.