First-Order Theories

CS 475

September 8, 2003

1 Preliminary

A *first-order theory* consists of an alphabet, a first order language, a set of axioms and a set of inference rules.

Definition 1 *An* alphabet *consists of the following sets:*

1. *variables*
2. *constants*
3. *function symbols*
4. *predicate symbols*
5. *connectives:* $\{\wedge, \vee, \neg, \leftrightarrow, \rightarrow\}$
6. *quantifiers:*$\forall, \exists$
7. *punctuation symbols:* $'(', ')', ',', '.'$

NOTE: • The last three sets are the same for every alphabet.

• For an alphabet, only the set of constants or the set of function symbols may be empty.

• Notation convention: Variables: u, v, w, x, y, and z (possibly with indexes); constants: a, b, and c (possibly with indexes); function symbols of arities > 0: f, g, and h (possibly with indexes); and predicate symbols of arities ≥ 0: p, q, and r (possibly with indexes).

The precedence among the connectives: $\neg, \forall, \exists, \wedge, \vee, \rightarrow, \leftrightarrow$

Given an alphabet, a *first order language* is defined by the set of *well-formed formula* (*wff* or *sentences*) of the theory (defined below).

Definition 2 *A* term *is either*

1. *a variable,*

2. *a constant, or*

3. *an expression of the form $f(t_1, \ldots, t_n)$ where f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms.*

Definition 3 *A* (well-formed) formula *is defined inductively as follows.*

1. *$p(t_1, \ldots, t_n)$ where p is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms,*

2. *if P and Q are formulas then $(\neg P)$, $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, $(P \leftrightarrow Q)$ are formulas*

3. *if P is a formula and x is a variable then $(\forall x\ P)$ and $(\exists x\ P)$ are formulas.*

Definition 4 *A first order language given by an alphabet consists of the set of all formulas constructed from the symbols of the alphabet.*

Example 1 *$(\forall x(\exists y(p(x, y) \rightarrow q(x))))$ and $(\neg\exists x((p(x, a) \wedge q(f(x)))))$ are formulas. We can simplify them to $\forall x\exists y(p(x, y) \rightarrow q(x))$ and $\neg\exists x(p(x, a) \wedge q(f(x)))$.*

Definition 5 *The scope of $\forall x$ (resp. $\exists x$) in $\forall xF$ (resp. $\exists xF$) is F. A* bound occurrence *of a variable in a formula is an occurrence immediately following a quantifier or an occurrence within the scope of the quantifier, which has the same variable immediately after the quantifier. Any other occurrence of a variable is* free.

Example 2 *$\exists xp(x, y) \rightarrow q(x)$ – the first two occurrences of x (in $\exists x$ and $p(x, y)$) are bound but the third one (in $q(x)$) is free.*

$\exists x(p(x, y) \rightarrow q(x))$ – all occurrences of x are bound (because of the parentheses!).

Definition 6 *A* closed formula *is a formula with no free occurrences of any variable.*

Example 3 *$\exists xp(x, y) \rightarrow q(x)$ is not a closed formula.*

$\exists x(p(x, y) \rightarrow q(x))$ is a closed formula.

Definition 7 *A* grounded term *is a term not containing a variable. A* grounded atom *is an atom not containing a variable.*

2 Interpretation

NOTE: When we say 'a first order language L' we understand that the alphabet of L is given.

Definition 8 *Let L be a first order language. An* interpretation *I of L consists of*

1. *a non-empty set D, called the* domain *of I,*

2. *for each constant in L, the assignment of an element of D, (i.e., a constant c is mapped into an element $I(c) \in D$),*

3. *for each n-ary function symbol in L, the assignment of a mapping from D^n to D, (i.e., a function symbol f is mapped into a function f^I)*

4. *for each n-ary predicate symbol in L, the assignment of a mapping from D^n to into $\{true, false\}$, (i.e., a predicate symbol p is mapped into a relation p^I).*

Let I be an interpretation. A *variable assignment (wrt. I)* is an assignment to each variable in L an element in D.

Let I be an interpretation and V be a variable assignment (wrt. I). The term assignment (wrt. I and V) of the terms in L is defined as follows.

1. Each variable is given its assignment according to V,

2. Each constant is given its assignment according to I,

3. If $t'_1, \ldots, t'_n$ are the term assignments of $t_1, \ldots, t_n$ and f' is the assignment of the n-ary function symbol f, then $f'(t'_1, \ldots, t'_n)$ is the term assignment of $f(t_1, \ldots, t_n)$.

Let I be an interpretation and V be a variable assignment (wrt. I). Then, a formula L can be given a *truth value*, true or false, (wrt. I and V) as follows:

1. If $L = p(t_1, \ldots, t_n)$ and $t'_1, \ldots, t'_n$ are the term assignments of $t_1, \ldots, t_n$ (wrt. I and V), and p' be the mapping assigned to the n-ary predicate symbol p by I, then the truth value of L is obtained by calculating the truth value of $p'(t_1, \ldots, t'_n)$,

2. If the formula has the form $(\neg P)$, $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, $(P \leftrightarrow Q)$ then the truth value of the formula is given by the following table

P	Q	$\neg$ P	P $\wedge$ Q	P $\vee$ Q	P $\rightarrow$ Q	P $\leftrightarrow$ Q
t	t	f	t	t	t	t
t	f	f	f	t	f	f
f	t	t	f	t	f	f
f	f	t	f	t	f	t

3. If the formula has the form $\exists x F$, then the truth value of the formula is true if there exists $d \in D$ such that F has the truth value wrt. I and the variable assignment V in which x is assigned to d; Otherwise, its truth value is false.

4. If the formula has the form $\forall x F$, then the truth value of the formula is true if for all $d \in D$, F has the truth value wrt. I and the variable assignment V in which x is assigned to d; Otherwise, its truth value is false.

From the above, the truth value of a closed formula does not depend on the variable assignment. Thus, we can speak about the truth value of a closed formula wrt. to an interpretation (without mentioning the variable assignment).

Definition 9 *A first order theory T is a set of formulas of a first order language L.*

Definition 10 *Let I be an interpretation of a first order language L and let F be a closed formula of L. Then, I is a* model *of F if F is true wrt. I.*

Let S be a closed formulas of a first order language L and I be an interpretation of L. I is a *model* of S if every formula $F \in S$ is true wrt. I.

Definition 11 *Let S be a closed formulas of a first order language L. We say*

1. *S is* satisfiable *if L has an interpretation which is a model of S,*
2. *S is* valid *if every interpretation of L is a model of S,*
3. *S is* unsatisfiable *if no interpretation of L is a model of S,*
4. *S is* nonvalid *if L has an interpretation which is not a model of S.*

Definition 12 *Let S be a closed formulas of a first order language L and F a formula in L. S entails F, denoted by $S \models F$, if F is true wrt. to every model of S.*

3 Unification

Definition 13 *A* literal *is either an atom P or its negation $\neg P$.*

Given a first order language L, an *expression* is either a constant, a variable, a term, a literal, a conjunction of literals, or a disjunction of literals.

Definition 14 *A* substitution *η is a finite set of the form $\{v_1/t_1, \ldots, v_n/t_n\}$ where each v_i is a variable, each t_i is a term distinct from v_i and the variables $v_1, \ldots, v_n$ are distinct.*

Each v_i/t_i is called a binding *for v_i;*

η is called a grounded substitution if the t_i are all ground terms.

Definition 15 *Let $\eta = \{v_1/t_1, \ldots, v_n/t_n\}$ be a substitution and E be an expression (term, literal, conjunction of literals, or disjunction of literals). Then, $E\eta$, the instance of E by η, is the expression obtained from E by simultaneously replacing each occurrence of the variables v_i in E by the term t_i.*

If $E\eta$ is ground, then it is called a ground instance *of E.*

Example 4 *Let $E = p(x, y, f(a))$ and $\eta = \{x/b, y/x\}$ then $E\eta = p(b, x, f(a))$.*

Definition 16 *Let $\sigma = \{v_1/t_1, \ldots, v_n/t_n\}$ and $\eta = \{u_1/s_1, \ldots, u_m/s_n\}$ be two substitutions. Then the* composition *$\eta\sigma$ of η and σ is the substitution obtained from the set $\{u_1/s_1\sigma, \ldots, u_m/s_n\sigma, v_1/t_1, \ldots, v_n/t_n\}$ by deleting any binding v_j/t_j for which $v_j \in \{u_1, \ldots, u_m\}$.*

Example 5 *Let $\sigma = \{x/a, y/b, z/y\}$ and $\eta = \{x/f(y), y/z\}$ then $\eta\sigma = \{x/f(b), z/y\}$ since $X = \{x/f(y)\sigma, y/z\sigma\} = \{x/f(b), y/y, x/a, y/b, z/y\}$ and $\eta\sigma$ is obtained from X by deleting $y/y, x/a, y/b$ because x and y are variables occurring in η.*

Two expressions E and F are called *variants* if there exists a substitution η such that $E = F\eta$ and $F = E\eta$. (We also say that E is a variant of F and vice versa!)

A set of expressions $S = \{E_1, \ldots, E_n\}$ is unifiable if there exists a substitution η such that $E_1\eta = E_2\eta = \ldots = E_n\eta$. In that case, η is a *unifier* of S.

A unifier η of S is called a *most general unifier* (or *mgu*) of S if for each unifier σ of S there exists a substitution γ such that $\sigma = \eta\gamma$.

Example 6 *The set $S = \{p(f(x), a), p(y, f(w))\}$ is not unifiable because we can not unify the constant a with $f(w)$.*

The set $S = \{p(f(x, y), g(z), a), p(f(y, x), g(u), a)\}$ is unifiable since for $\eta = \{x/y, z/u\}$, $S\eta = \{p(f(y, y), g(u), a)\}$. Here, η is a mgu of S.

Definition 17 *Let S be a set of simple expressions (a simple expression is a term or an atom). The* disagreement set *of S is defined as follows. Locate the leftmost symbol position at which not all expressions in S have the same symbol and extract from each expression expression in S the subexpression beginning at that symbol position. The set of all such subexpressions is the disagreement set.*

Example 7 *Let $S = \{p(f(x), h(y), a), p(f(x), z, a), p(f(x), h(y), b)\}$. Then the disagreement set is $\{h(y), z)\}$.*

Example 8 *Let $S = \{p(f(x), h(y), a), p(f(x), z, a), p(f(x), w, b)\}$. Then the disagreement set is ?.*

4 Unification Algorithm

Let $S = \{P_1, \ldots, P_m\}$ be a set of simple expressions.

S1 Put $k = 0$ and $\sigma_0 = \{\}$.

S2 If $S\sigma_k$ is a singleton ($P_i\sigma_k = P_j\sigma_k$ for every $i \neq j$), then stop; σ_k is an mgu (most general unifier) of S; Otherwise, find the disagreement set D_k of $S\sigma_k$.

S3 If there exist v and t in D_k such that v is a variable that does not occur in t, then put $\sigma_{k+1} = \sigma_k\{v/t\}$, increment k and go to S2. Otherwise, stop; S is not unifiable.

Example 9 *Let* $S = \{p(f(a), g(x)), p(y, y)\}$.

S1 *Put* $k = 0$ *and* $\sigma_0 = \{\}$.

S2 $S\sigma_0 = S$ *is not a singleton. So, we need to find the disagreement set* D_0 *of* $S\sigma_0 = S$. *We have:* $D_0 = \{f(a), y\}$.

S3 *Here,* y *is a variable which does not occur in* $f(a)$. *So, we let* $\sigma_1 = \sigma_0\{y/f(a)\} = \{y/f(a)\}$ *and go to S2.*

S2 $S\sigma_1 = \{p(f(a), g(x)), p(f(a), f(a))\}$ *is not a singleton. So, we need to find the disagreement set* D_1 *of* $S\sigma_1 = S$. *We have:* $D_1 = \{g(x), f(a)\}$.

S3 *Here, there is no variable in* D_1. *So, we stop;* S *is not unifiable.*

Example 10 *Let* $S = \{p(a, x, h(g(z))), p(z, h(y), h(y))\}$.

S1 *Put* $k = 0$ *and* $\sigma_0 = \{\}$.

S2 $S\sigma_0 = S$ *is not a singleton. So, we need to find the disagreement set* D_0 *of* $S\sigma_0 = S$. *We have:* $D_0 = \{a, z\}$.

S3 *Here,* z *is a variable which does not occur in* a. *So, we let* $\sigma_1 = \sigma_0\{z/a\} = \{z/a\}$ *and go to S2.*

S2 $S\sigma_1 = \{p(a, x, h(g(a))), p(a, h(y), h(y))\}$ *is not a singleton. So, we need to compute the disagreement set* D_1 *of* $S\sigma_1$. *We have:* $D_1 = \{x, h(y)\}$.

S3 *Here,* x *is a variable which does not occur in* $h(y)$. *So, we let* $\sigma_2 = \sigma_1\{x/h(y)\} = \{z/a, x/h(y)\}$ *and go to S2.*

S2 $S\sigma_2 = \{p(a, h(y), h(g(a))), p(a, h(y), h(y))\}$ *is not a singleton. So, we need to find the disagreement set* D_2 *of* $S\sigma_2$. *We have:* $D_1 = \{y, g(a)\}$.

S3 *Here,* y *is a variable which does not occur in* $g(a)$. *So, we let* $\sigma_3 = \sigma_2\{y/g(a)\} = \{z/a, x/h(g(a)), y/g(a)\}$ *and go to S2.*

S2 $S\sigma_3 = \{p(a, h(g(a)), h(g(a)))\}$ *is a singleton. So we stop and one mgu of* S *is* $\sigma_3 = \{z/a, x/h(g(a)), y/g(a)\}$.

Theorem 1 *Let* S *be a finite of simple expressions. If* S *is unifiable then the algorithm terminates and gives an mgu for* S. *If* S *is not unifiable then the algorithm terminates and reports this fact.*

5 Resolution

Definition 18 *A* literal *is either an atom P or its negation $\neg P$.*

A clause *is a disjunction of literals. (sometime it is written as $P_1 \vee \ldots \vee P_n$ or $\{P_1, \ldots, P_n\}$)*

A formula Q is said to be in conjunctive normal form *(or* CNF*) if Q is a conjunction of clauses.*

A formula Q is said to be in implicative normal form *if Q is a conjunction of implication of the form $P_1 \wedge \ldots \wedge P_n \rightarrow Q_1 \vee \ldots \vee Q_m$ where each P_i, Q_j is an atom.*

$(P \vee Q \vee \neg S) \wedge (\neg P \vee Q \vee S) \wedge (\neg P \vee \neg R \vee \neg S) \wedge (P \vee T \vee \neg S)$ is a CNF.

Theorem 2 *For every formula ϕ there exists a formula ψ in CNF form such that ϕ and ψ is equivalent, i.e., $\forall(\phi \leftrightarrow \psi)$ is a valid formula.*

Algorithm to convert a formula into CNF form.

1. **Eliminate implications**: Replace $p \rightarrow q$ with $\neg p \vee q$
2. **Move $\neg$ inward**: do the following
   (a) $\neg(p \vee q)$ is replaced by $\neg p \wedge \neg q$
   (b) $\neg(p \wedge q)$ is replaced by $\neg p \vee \neg q$
   (c) $\neg \forall x p$ is replaced by $\exists x p$
   (d) $\neg \exists x p$ is replaced by $\forall x \neg p$
   (e) $\neg\neg p$ is replaced by p
3. **Standardize variable**: For sentences like $(\forall x P(x)) \vee (\exists x Q(x))$ that use the same variable name twice, change the name of one of the variable.
4. **Move quantifier left**: Move all quantifiers in the formula to the left, for example, $p \vee \forall x q$ is equivalent to $\forall x q \vee p$ etc.
5. **Skolemize**: Remove the existential quantifier by elimination – this includes: (1) defines a Skolem function, one for a variable occurred immediately after an existential quantification, (2) introduces a new constant, one for a variable occurred immediately after an existential quantification, (3) removes the existential quantification and substitutes x for $F^x(A^x)$ in the formula.
6. **Distribute $\wedge$ over $\vee$**: $(a \wedge b) \vee c$ becomes $(a \vee c) \wedge (b \vee c)$.
7. **Flatten nested conjunction and disjunction**: $(a \wedge b) \wedge c$ becomes $(a \wedge b \wedge c)$ and $(a \vee b) \vee c$ becomes $(a \vee b \vee c)$.

Example 11 *Convert* $((\neg\forall x A(x)) \vee (\forall y B(y))) \rightarrow (\neg(\forall z Q(z, f(z))))$ *to CNF.*

1. $\neg((\neg\forall x A(x)) \vee (\forall y B(y))) \vee (\neg(\forall z Q(z, f(z))))$ *(Eliminate implication)*

2. $((\neg\neg\forall x A(x)) \wedge (\neg\forall y B(y))) \vee (\neg(\forall z Q(z, f(z))))$ *(Move* $\neg$ *...)*

3. $(\forall x A(x) \wedge \exists y \neg B(y)) \vee ((\exists z \neg Q(z, f(z))))$ *(Move* $\neg$ *...)*

4. $\forall x \exists y \exists z ((A(x) \wedge \neg B(y)) \vee \neg Q(z, f(z)))$ *(Move quantifier left)*

5. $\forall x ((A(x) \wedge \neg B(Fy(Cy))) \vee \neg Q(Cz, f(Fz(Cz))))$ *(Removing existential quantifier -* Fy, Fz *are two new functions and* Cy, Cz *are two new constants, correspond to the variable* y *and* z *respectively)*

6. $(A(x) \wedge \neg B(Fy(Cy))) \vee \neg Q(Cz, f(Fz(Cz)))$ *(Drop universal quantifier)*

7. $(A(x) \vee \neg Q(Cz, f(Fz(Cz)))) \wedge (\neg B(Fy(Cy)) \vee \neg Q(Cz, f(Fz(Cz))))$ *(Distribute* $\wedge$ *over* $\vee$*)*

NOTE: Implicative normal form is often used to. A formula of the form $\neg P_1 \vee \neg P_2 \vee \ldots \vee \neg P_n \vee Q_1 \vee Q_2 \ldots \vee Q_m$ is equivalent to $P_1 \wedge P_2 \ldots \wedge P_n \rightarrow Q_1 \vee Q_2 \ldots \vee Q_m$

It is easy to see that if Q is in CNF then we can convert it into implicative normal form using the above conversion.

The *resolution inference rule* If β_1 and β_2 are unifiable and η is a mgu of β_1 and β_2, then

$$\frac{\alpha \vee \beta_1, \neg\beta_2 \vee \gamma}{\alpha\eta \vee \gamma\eta} \tag{1}$$

or

$$\frac{\neg\alpha \rightarrow \beta_1, \beta_2 \rightarrow \gamma}{\neg\alpha\eta \rightarrow \gamma\eta} \tag{2}$$

We can
Given a set of formulas S and a formula Q, we would like to determines if $S \models Q$.

We can use (1) (or (2)) to determine whether $S \vdash Q$ holds or not.

We make the following assumptions:

1. Each formula in S is a clause (**why?**)
2. Q is a literal (**why?**)

Example 12 *Let* Δ *be the set consisting of the following clauses:*

1. $\neg P(w) \vee Q(w)$*,*

2. $P(x) \vee R(x)$*,*

3. $\neg Q(y) \vee S(y)$*, and*

4. $\neg R(z) \vee S(z)$.

Question: $\Delta \vdash S(A)$*?*

Proof.

1. $\frac{\neg P(w) \vee Q(w), P(x) \vee R(x)}{\neg P(w) \vee S(w)}$ *where* $\eta = \{y/w\}$

2. $\frac{\neg P(w) \vee S(w), P(x) \vee R(x)}{S(x) \vee R(x)}$ *with* $\{w/x\}$

3. $\frac{S(x) \vee R(x), \neg R(z) \vee S(z)}{S(A)}$ *with* $\{x/A, z/A\}$*! DONE!*

Refutation proof procedure. Given a set of clauses S and a literal Q. The refutation proof procedure uses resolution to determine whether $S \models Q$ holds or not.

1. **Idea:** If $S \models Q$ then $S \cup \{\neg Q\}$ is unsatisfiable, i.e., there is no model for $S \cup \{\neg Q\}$. So, we will assume that $\neg Q$ holds and try to derive a contradiction out of $S \cup \{\neg Q\}$.

2. **Algorithm:** We try to derive a proof that derives a contradiction from $S \cup \{\neg Q\}$. The algorithm can be described as follows.

   A1 Let $k = 0$, $G_k = \neg Q$.

   A2 If $G_k = false$ then step and answer 'yes'; Otherwise, find a clause C in S that contains a literal L which is contradictory with some L' of G_k and η is a mgu of L and L'. Go to step [A3]!

   A3 Let $G_{k+1} = ((C \setminus \{L\}) \cup (G_k \setminus \{L'\}))\eta$, $k = k + 1$, and go to step [A2]!

Example 13

$$Dog(D) \tag{3}$$
$$Owns(Jack, D) \tag{4}$$
$$Dog(y) \wedge Owns(x, y) \rightarrow AnimalLover(x) \tag{5}$$
$$AnimalLover(x) \wedge Animal(y) \wedge Kills(x, y) \rightarrow False \tag{6}$$
$$Kills(Jack, Tuna) \vee Kills(Curiosity, Tuna) \tag{7}$$
$$Cat(Tuna) \tag{8}$$
$$Cat(x) \rightarrow Animal(x) \tag{9}$$

Convert to clausal form

$$Dog(D) \tag{10}$$
$$Owns(Jack, D) \tag{11}$$
$$\neg Dog(y) \vee \neg Owns(x, y) \vee AnimalLover(x) \tag{12}$$
$$\neg AnimalLover(x) \vee \neg Animal(y) \vee \neg Kills(x, y) \tag{13}$$
$$Kills(Jack, Tuna) \vee Kills(Curiosity, Tuna) \tag{14}$$
$$Cat(Tuna) \tag{15}$$
$$\neg Cat(x) \vee Animal(x) \tag{16}$$

Proving $Kills(Curiosity, Tuna)$

$G_0 = \neg Kills(Curiosity, Tuna)$, $\eta = \{\}$, *Clause (14)*

$G_1 = Kills(Jack, Tuna)$, $\eta = \{x/Jack, y/Tuna\}$, *Clause (13)*

$G_2 = \neg AnimalLover(Jack) \vee \neg Animal(Tuna)$, $\eta = \{x/Tuna\}$, *Clause (16)*

$G_3 = \neg AnimalLover(Jack) \vee \neg Cat(Tuna)$, $\eta = \{x/Tuna\}$, *Clause (16)*

$G_4 = \neg AnimalLover(Jack)$, $\eta = \{\}$, *Clause (15)*

$G_5 = \neg Dog(y) \vee \neg Owns(Jack, y)$, $\eta = \{x/Jack\}$, *Clause (12)*

$G_6 = \neg Dog(D)$, $\eta = \{y/D\}$, $\eta = \{\}$, *Clause (11)*

$G_7 = \square$ *(or* $G_7 = false$*)*, $\eta = \{\}$, *Clause (10)! DONE*

Remark. The steps can be drawn into a *refutation tree* as follows: