Loop Formulas for Disjunctive Logic Programs

http://www.cs.utexas.edu/~appsmurf/disjunctive.ps

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Background


- Under “tightness”, two semantics are equivalent.

- Lin/Zhao [2002] introduced the concept of a loop formula. Answer sets for a nondisjunctive program can be characterized as the models of its completion that satisfy the loop formulas.

- SAT-based answer set solvers: ASSAT, CMODELS.
Our Contribution

- We extend Clark’s completion and Lin/Zhao’s work on loop formulas to disjunctive logic programs. This explains a puzzling feature of Lin/Zhao’s work.

- The concept of a tight program and Fages’ theorem are extended to disjunctive programs as well.

- SAT-based answer set solvers can be extended to deal with disjunctive programs.
Completion

We assume that every rule of $\Pi$ has the form

$$A \leftarrow Body$$

where $A$ is a clause (disjunction of distinct atoms).

The completion of $\Pi$, $\text{Comp}(\Pi)$, consists of

$$Body \supset A$$

for every rule $A \leftarrow Body$ in $\Pi$, and

$$a \supset \bigvee_{A\leftarrow Body \in \Pi} \left( Body \land \bigwedge_{p \in A \setminus \{a\}} \neg p \right)$$

for each atom $a$. 
Examples

\( \Pi_1 : \ p ; q \)
\( \text{Comp}(\Pi_1) : \ p \lor q \)
\( p \triangleright q \)
\( q \triangleright \neg p \)

A.S. : \{p\}, \{q\}
Models : \{p\}, \{q\}

\( \Pi_2 : \ p ; q \)
\( p \leftarrow q \)
\( q \leftarrow p \)
\( \text{Comp}(\Pi_2) : \ p \lor q \)
\( q \triangleright p \)
\( p \triangleright q \)
\( p \triangleright \neg q \lor q \)
\( q \triangleright \neg p \lor p \)

A.S. : \{p, q\}
Models : \{p, q\}
\[ \Pi_3 : \quad p ; r \leftarrow q \]
\[ \quad q \leftarrow p \]
\[ \quad p \leftarrow \text{not } r \]
\[ \quad r \leftarrow r \]

\[ \text{Comp}(\Pi_3) : \quad q \supset p \lor r \]
\[ \quad p \supset q \]
\[ \quad \neg r \supset p \]
\[ \quad r \supset r \]
\[ \quad p \supset (q \land \neg r) \lor \neg r \]
\[ \quad q \supset p \]
\[ \quad r \supset (q \land \neg p) \lor r \]

A.S. : \{p, q\}  
Models : \{p, q\}, \{r\}

**Proposition 1** For any program \( \Pi \) and any set \( X \) of atoms, if \( X \) is an answer set for \( \Pi \), then \( X \) is a model of \( \text{Comp}(\Pi) \).
Positive Dependency Graph

The *positive dependency graph* of $\Pi$ is the directed graph $G$ such that

- the vertices of $G$ are the atoms occurring in $\Pi$, and
- for every rule $A \leftarrow Body$ in $\Pi$, $G$ has an edge from each atom in $A$ to each atom in $pa(Body)$.

$pa(Body)$ : the set of its “positive atoms”. the set of all atoms $a$ such that at least one occurrence of $a$ in $Body$ is not in the scope of negation as failure.
\[ \Pi_1 : \ p \; ; \; q \quad \Pi_2 : \ p \; ; \; q \quad \Pi_3 : \ p \; ; \; r \leftarrow q \]

\[
\begin{align*}
p & \leftarrow q \\
q & \leftarrow p \\
p & \leftarrow not \; r \\
r & \leftarrow r
\end{align*}
\]

Loops :

- no loops
- \( \{p, q\} \)
- \( \{p, q\}, \{r\} \)
Loop Formulas

- We assume that every rule of $\Pi$ has the form
  \[ A \leftarrow B, F \]
  where $A$ is a clause, $B$ is a list of atoms; every occurrence of each atom in $F$ is in the scope of negation as failure.
- For any loop $L$ of $\Pi$, by $R(L)$ we denote the set of formulas
  \[ B \land F \land \bigwedge_{p \in A \setminus L} \neg p \]
  for all rules $A \leftarrow B, F$ in $\Pi$ such that $A \cap L \neq \emptyset$ and $B \cap L = \emptyset$.

\[ \Pi_2 : \begin{align*}
  & p ; q \\
  & p \leftarrow q \\
  & q \leftarrow p
\end{align*} \]

\[ \Pi_3 : \begin{align*}
  & p ; r \leftarrow q \\
  & q \leftarrow p \\
  & p \leftarrow \text{not } r \\
  & r \leftarrow r
\end{align*} \]

\[ \Pi_2 : R(\{p, q\}) : \{ \top \} \]

\[ \Pi_3 : R(\{p, q\}) : \{ \neg r \}, \ R(\{r\}) : \{ q \land \neg p \} \]
Loop Formulas—cont’d

$CLF(L)$: the conjunctive loop formula for $L$:
$$CLF(L) = \bigwedge L \supset \bigvee R(L).$$

$CLF(\Pi)$: the set of all conjunctive loop formulas for $\Pi$:
$$CLF(\Pi) = \{ CLF(L) : L \text{ is a loop of } \Pi \}.$$

$\Pi_2 : R(\{p, q\}) : \{\top\}$

$CLF(\Pi_2) = \{p \land q \supset \top\}$

$\Pi_3 : R(\{p, q\}) : \{\neg r\}, \ R(\{r\}) : \{q \land \neg p\}$

$CLF(\Pi_3) = \{p \land q \supset \neg r, \ r \supset q \land \neg p\}$

**Theorem 1** For any program $\Pi$ and any set $X$ of atoms, $X$ is an answer set for $\Pi$ iff $X$ is a model of $\text{Comp}(\Pi) \cup CLF(\Pi)$. 
Absolutely Tight Program

A program is *absolutely tight* if it has no loops.

**Corollary 1** For any absolutely tight program $\Pi$ and any set $X$ of atoms, $X$ is an answer set for $\Pi$ iff $X$ satisfies $\text{Comp}(\Pi)$.

$\Pi_1: \ p ; q$

$\text{Comp}(\Pi_1): \ p \lor q$

$p \supset \neg q$

$q \supset \neg p$

A.S.: $\{p\}, \{q\}$

Models: $\{p\}, \{q\}$