Sets

- A set is a collection of objects
- We will mostly use sets of like (abstract) objects – a domain
- The elements of a set are written between braces and separated by commas
- E.g. the natural numbers, usually written as $\mathbb{N} = \{0, 1, 2, \ldots\}$ which is an infinite set

Operations on sets

- Element-of or member-of: $x \in S$ if $x$ is a member of the set $S$
- Subset: $S \subseteq T \iff \forall x. x \in S \Rightarrow x \in T$, i.e. if the element is in $S$, then it’s in $T$
- Proper subset: $S \subset T \iff \forall x. x \in S \Rightarrow x \in T \land \exists y. y \in T \land y \not\in S$, i.e. there is at least one element of $T$ that is not in $S$
Operations on sets

- Union: $A \cup B = \{x | x \in A \lor x \in B\}$
- Intersection: $A \cap B = \{x | x \in A \land x \in B\}$
- Difference: $A - B = \{x | x \in B \land x \notin A\}$

E.g. if $A = \{a, c, e, g\}$, $B = \{d, e, f, g\}$, then
  - $A \cup B = \{a, c, d, e, f, g\}$
  - $A \cap B = \{e, g\}$
  - $A - B = \{a, c\}$
  - $B - A = \{d, f\}$

Operations on sets

- Grand union (of a sets of sets):
  $\bigcup S = \{x | \exists T \in S. x \in T\}$
- Grand intersection (of a set of sets):
  $\bigcap S = \{x | \forall T \in S. x \in T\}$

Relations

- Two sets have a relation between them if there exists a set of pairs of elements taken from each of the sets
- We can write a pair as $[x, y]$ where, for two sets $S$ and $T$, $x \in S$ and $y \in T$ ($S$, called the domain and $T$, called the range, could be the same set, but not necessarily)
- The set of a number of such pairs is a relation
- We can write, for a relation $\rho$, $[x, y] \in \rho$, or, equivalently, $x \rho y$
- Domain of $\rho \triangleq \{x | \exists y. x \rho y\}$
- Range of $\rho \triangleq \{y | \exists x. x \rho y\}$

Picture of a relation

Notice that in a general relation, any element in the domain can be paired with any element in the range, and some can be omitted from any pair. Placing different restrictions on what can be paired with what yields relations with different properties (symmetry, reflexiveness, transitivity). We shall mainly be concerned with the special relation called a function.
Relations

- The identity relation for a set $S$ is:
  \[ I_S = \{ (x, x) \mid x \in S \} \]

- The universal relation is all possible pairings. It is written as $S \times T$, called the Cartesian product.

- Any arbitrary relation between two sets is a subset of this relation:
  \[ \{ (x, y) \mid x \in S \land y \in T \} \subseteq S \times T \]

- Relations need not be binary:
  \[ \{ (x_1, x_2, \ldots, x_n) \mid x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n \} \subseteq S_1 \times S_2 \times \cdots \times S_n \]

- Shorthand:
  \[ \prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n \]