Assignment 4: Denotational Semantics

Turn in your answer through the submission page (click on the link on the Assignments page) by 5:00pm on Wednesday November 2nd.

The denotational semantics of both binary numerals and Schmidt’s calculator are at the end of this paper. Refer to those definitions to answer the following questions.

1. Use the binary numeral semantics as given to determine the meanings of the following derivation trees:
   a. [0011]
   b. [000]
   c. [111]

2. a. In a fashion similar to that given in the semantics of binary numerals, define a denotational semantics for the language of base 8 numerals, Octal. Let E be the valuation function:
   \[ E : \text{Octal} \to \text{Nat}. \]
   b. Prove the following equivalence: \[ E[[015]] = B[[1101]]. \]

3. Simplify these calculator programs to their meanings in Nat*:
   a. \[
   \text{ON } 1 + ( \text{IF \ LASTANSWER, 4, 1) TOTAL \ LASTANSWER TOTAL 5 * 2 TOTAL OFF}}
   \]
   b. \[
   \text{ON 5 TOTAL 5 TOTAL 10 TOTAL OFF}}
   \]

4. Alter the calculator semantics as given so that the memory cell argument to S and E becomes a memory stack; that is, use Nat* in place of Nat as an argument domain to S and E. (Hint: Nat* is a list, but the operation \text{hd} is like \text{pop} and \text{cons} is like \text{push}.)

**Binary Numerals**

Abstract syntax:

B ∈ Binary-numeral
D ∈ Binary-digit
B ::= BD | D
D ::= 0 | 1

Semantic algebras:

Domain $Nat = \mathbb{N}$

Operations

$zero, one, two, \ldots : Nat$

$plus, times : Nat \rightarrow Nat \rightarrow Nat$

Valuation functions:

$B : Binary-numeral \rightarrow Nat$

$B[BD] = (B[B] \text{ times two}) plus D[D]$

$B[D] = D[D]$

$D : Binary-digit \rightarrow Nat$

$D[0] = zero$

$D[1] = one$

**The Calculator**

Abstract syntax:

P ∈ Program
S ∈ Expression-sequence
E ∈ Expression
N ∈ Numeral
D ∈ Digit

P ::= ON S
S ::= E TOTAL S | E TOTAL OFF
E ::= E₁ + E₂ | E₁ * E₂ | N | ( E ) | IF E₁ , E₂ , E₃ | LASTANSWER
N ::= N D | D
D ::= 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 0

Semantic algebras:

Domain: List = $A^*$, where $A$ is any domain

Operations:
\[\text{Domain: } \mathbb{N} = \mathbb{N}\]

\text{Operations: }
\begin{align*}
\text{zero, one, two, \ldots: } & \mathbb{N} \\
\text{plus: } & \mathbb{N} \rightarrow \mathbb{N} \\
\text{times: } & \mathbb{N} \rightarrow \mathbb{N} \\
\text{equals: } & \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{B}
\end{align*}

\text{Domain: } \mathbb{B} = \mathbb{B}

\text{Operations: }
\begin{align*}
\text{true, false: } & \mathbb{B} \\
\rightarrow & \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{A}, \text{ for any domain } \mathbb{A}.
\end{align*}

\text{Valuation functions: }
\begin{align*}
P: \text{Program} & \rightarrow \mathbb{N}^* \\
S: \text{Expression-sequence} & \rightarrow \mathbb{N} \rightarrow \mathbb{N}^* \\
E: \text{Expression} & \rightarrow \mathbb{N} \rightarrow \mathbb{N}
\end{align*}

\text{N: Numeral} \rightarrow \mathbb{N}
\[ N[N D] = (\text{ten times } N[N]) + D[D] \]
\[ N[D] = D[D] \]