Quantifiers in First Order Logic

List of Logical Equivalences

- \( p \land T \iff p \) (Identity Laws)
- \( p \lor F \iff p \) (Identity Laws)
- \( p \lor T \iff T \) (Domination Laws)
- \( p \land F \iff F \) (Domination Laws)
- \( p \lor p \iff p \) (Idempotent Laws)
- \( p \land p \iff p \) (Idempotent Laws)
- \( \neg(\neg p) \iff p \) (Double Negation Law)
- \( p \lor q \iff q \lor p \) (Commutative Laws)
- \( p \land q \iff q \land p \) (Commutative Laws)
- \( (p \lor q) \lor r \iff p \lor (q \lor r) \) (Associative Laws)
- \( (p \land q) \land r \iff p \land (q \land r) \) (Associative Laws)
- \( p \lor q \iff q \lor p \) (Commutative Laws)
- \( p \land q \iff q \land p \) (Commutative Laws)
- \( (p \lor q) \lor r \iff p \lor (q \lor r) \) (Associative Laws)
- \( (p \land q) \land r \iff p \land (q \land r) \) (Associative Laws)

List of Equivalences

- \( p \lor (q \land r) \iff (p \lor q) \land (p \lor r) \) (Distribution Laws)
- \( p \land (q \lor r) \iff (p \land q) \lor (p \land r) \) (Distribution Laws)
- \( \neg(p \lor q) \iff \neg p \land \neg q \) (De Morgan's Laws)
- \( \neg(p \land q) \iff \neg p \lor \neg q \) (De Morgan's Laws)
- \( p \lor \neg p \iff T \) (Or Tautology)
- \( p \land \neg p \iff F \) (And Contradiction)
- \( (p \land q) \land r \iff (p \land q) \land r \) (Implication Equivalence)
- \( (p \lor q) \lor r \iff (p \lor q) \lor r \) (Implication Equivalence)
- \( (p \land q) \land r \iff p \land (q \land r) \) (Biconditional Equivalence)
- \( (p \lor q) \lor r \iff (p \lor q) \lor r \) (Biconditional Equivalence)

The Proof Process

- **Assumptions**
- **Logical Steps**
  - Definitions
  - Already-proved equivalences
  - Statements (e.g., arithmetic or algebraic)
- **Conclusion**
  - (That which was to be proved)

Predicate Calculus: Quantifiers

**Universe of Discourse, U:** The domain of a variable in a propositional function.

**Universal Quantification** of \( P(x) \) is the proposition: “\( P(x) \) is true for all values of \( x \) in \( U \).”

**Existential Quantification** of \( P(x) \) is the proposition: “There exists an element, \( x \), in \( U \) such that \( P(x) \) is true.”

**Prove:** \( (p \land \neg q) \lor q \iff p \lor q \)

\( (p \land \neg q) \lor q \iff \) \( q \lor (p \land \neg q) \) \( \iff \) Left-Hand Statement
\( q \lor (p \land \neg q) \iff \) \( q \lor (p \land \neg q) \) \( \iff \) Commutative
\( q \lor (p \land \neg q) \iff \) \( (q \lor p) \land q \lor (q \lor p) \) \( \iff \) Distributive
\( (q \lor p) \land q \lor (q \lor p) \iff \) \( (q \lor p) \land q \lor (q \lor p) \) \( \iff \) Or Tautology
\( (q \lor p) \land q \lor (q \lor p) \iff \) \( q \lor p \) \( \iff \) Identity
\( q \lor p \iff \) \( q \lor p \) \( \iff \) Commutative
\( q \lor p \iff \) \( q \lor p \) \( \iff \) Commutative

Begin with exactly the left-hand side statement
End with exactly what is on the right
Justify EVERY step with a logical equivalence
Universal Quantification of $P(x)$

$\forall x P(x)$

“for all $x$ $P(x)$”

“for every $x$ $P(x)$”

Defined as:

$P(x_0) \land P(x_1) \land P(x_2) \land P(x_3) \land \ldots \text{ for all } x_i \text{ in } U$

Example:

Let $P(x)$ denote $x^2 \geq x$

If $U$ is $x$ such that $0 < x < 1$ then $\forall x P(x)$ is false.

If $U$ is $x$ such that $1 < x$ then $\forall x P(x)$ is true.

Existential Quantification of $P(x)$

$\exists x P(x)$

“there is an $x$ such that $P(x)$”

“there is at least one $x$ such that $P(x)$”

“there exists at least one $x$ such that $P(x)$”

Defined as:

$P(x_0) \lor P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots \text{ for all } x_i \text{ in } U$

Example:

Let $P(x)$ denote $x^2 \geq x$

If $U$ is $x$ such that $0 < x \leq 1$ then $\exists x P(x)$ is true.

If $U$ is $x$ such that $x < 1$ then $\exists x P(x)$ is true.

Quantifiers

$\forall x P(x)$

• True when $P(x)$ is true for every $x$.

• False if there is an $x$ for which $P(x)$ is false.

$\exists x P(x)$

• True if there exists an $x$ for which $P(x)$ is true.

• False if $P(x)$ is false for every $x$.

Negation (it is not the case)

$\neg \exists x P(x)$ equivalent to $\forall x \neg P(x)$

• True when $P(x)$ is false for every $x$

• False if there is an $x$ for which $P(x)$ is true.

$\neg \forall x P(x)$ is equivalent to $\exists x \neg P(x)$

• True if there exists an $x$ for which $P(x)$ is false.

• False if $P(x)$ is true for every $x$.

Examples 2a

Let $T(a,b)$ denote the propositional function “a trusts b.” Let $U$ be the set of all people in the world.

Everybody trusts Bob.

$\forall x T(x, Bob)$

Could also say: $\forall x \in U \ T(x, Bob)$

$\in$ denotes membership

Bob trusts somebody.

$\exists x T(Bob, x)$

Examples 2b

Alice trusts herself.

$T(Alice, Alice)$

Alice trusts nobody.

$\forall x \neg T(Alice, x)$

Carol trusts everyone trusted by David.

$\forall x (T(David, x) \rightarrow T(Carol, x))$

Everyone trusts somebody.

$\forall x \exists y T(x, y)$
Quantification of Two Variables
(read left to right)

∀x∀yP(x,y) or ∀y∀xP(x,y)
• True when P(x,y) is true for every pair x,y.
• False if there is a pair x,y for which P(x,y) is false.

∃x∃yP(x,y) or ∃y∃xP(x,y)
True if there is a pair x,y for which P(x,y) is true.
False if P(x,y) is false for every pair x,y.

Examples 3a
Let L(x,y) be the statement “x loves y” where U for both x and y is the set of all people in the world.
Everybody loves Jerry.
∀xL(x,Jerry)

Everybody loves somebody.
∀x ∃yL(x,y)

There is somebody whom everybody loves.
∃y ∀xL(x,y)

Examples 3b1
There is somebody whom Lydia does not love.
∃x ¬L(Lydia,x)

Nobody loves everybody. (For each person there is at least one person they do not love.)
∀x ∃y ¬L(x,y)

There is somebody (one or more) whom nobody loves.
∃y ∀x ¬L(x,y)