Theoretical Foundations of Logic Programming

Mirosław Truszczyński

Department of Computer Science
University of Kentucky

July 24-27, 2008

Introduction

Logic programming

What is it?

- Declarative programming formalizm
- Knowledge representation formalizm

Two facets

- Prolog
- Answer-set programming

Logic programming

What is it?

- Declarative programming formalizm
- Knowledge representation formalizm

Two facets

- Prolog
- Answer-set programming

My goal

To present foundations of LP

- Focus on negation and its semantics

Overview

Roughly ...

- Basic syntax and semantics
- Horn logic programming — basis for Prolog (briefly)
- The need for negation
- Semantics of negation (supported, stable, Kripke-Kleene, well-founded)
- Properties of semantica (completion, Fages Lemma, loop theorem, equivalence)
- More general settings (logic HT, algebra)
- (Some) proofs

Some logic terminology

Language

- Constant, variable, function and predicate symbols
- Terms: strings built recursively from constant, variable and function symbols
- $c, X, f(c, X), f(f(c, X), f(X, f(X, c)))$
- Atoms: built of predicate symbols and terms
- $p(X, c, f(a, Y))$

Horn logic programming

Horn clause

- $p \leftarrow q_1, \ldots, q_k$
  - where p, q_i are atoms
- Clauses are universally quantified
  - special sentences
- Intuitive reading: if $q_1, \ldots, q_k$ then p

Horn program

- A collection of Horn clauses

Horn logic programming

Horn clause

- $p \leftarrow q_1, \ldots, q_k$
  - where p, q_i are atoms
- Clauses are universally quantified
  - special sentences
- Intuitive reading: if $q_1, \ldots, q_k$ then p

Horn program

- A collection of Horn clauses

More terminology

Herbrand model

- Ground terms: no variable symbols
- Herbrand universe: collection of all ground terms
- Ground atoms: atoms built of predicate symbols and ground terms
- $p(a, c, f(a, a))$
- Herbrand base: collection of all ground atoms
- Herbrand model: subset of an Herbrand base

Horn logic programming

Semantics

- Given by Herbrand models
  - $grnd(P)$: the set of all ground instances of clauses in P
  - Thus, no difference between P and $grnd(P)$
- Key question:
  which ground facts hold in every Herbrand model of P?
- Sufficient to restrict to Herbrand models contained in $HB(P)$
  - Herbrand universe of P, $HU(P)$
    (if no constant symbols in P, a single constant symbol introduced)
  - Herbrand base of P, $HB(P)$
  - Ground atoms not in $HB(P)$ are not true in all Herbrand models

We can say more

Least Herbrand model

- Every Horn program P has a least Herbrand model $LM(P)$
  - the intersection of a set of Herbrand models of a Horn program is a Herbrand model of the program
  - $HB(P)$ is an Herbrand model of P
  - the intersection of all models is a least Herbrand model (and it is contained in $HB(P)$)
- Single intended Herbrand model
- For a ground t, $P \models p(t)$ if and only if $p(t) \in LM(P)$
- For ground t, if $P \not\models p(t)$, defeasibly conclude $\neg p(t)$
- Closed World Assumption (CWA)

Computing with Horn programs

What do they specify, what can they express?

- A Horn program P specifies a subset X of the Herbrand universe for P, $HU(P)$, if for some predicate symbol p occurring in P we have:
$$X = \{t \in HU(P) \colon p(t) \in LM(P)\}$$
- Finite Horn programs specify precisely the r.e. sets and relations
Smullyan, 1968, Andreka and Nemeti, 1978

Reachability — an example

Program P

$$arc(a, b).$$
$$arc(b, c).$$
$$arc(d, c).$$

$$reach(X, X).$$
$$reach(X, Y) \leftarrow arc(X, Z), reach(Z, Y).$$

Reachability — an example

$HU(P)$, $HB(P)$, $ground(P)$

- $HU(P) = \{a, b, c, d\}$
- $HB(P) = \{arc(a,a), arc(a,b), \ldots, reach(a,a), \ldots\}$
- $ground(P)$:

$$
\begin{array}{l}
arc(a,b). \quad arc(b,c). \quad arc(d,c). \\
reach(a,a). \quad reach(b,b). \quad reach(c,c). \quad reach(d,d). \\
reach(a,a). \leftarrow arc(a,a), reach(a,a). \\
reach(a,b). \leftarrow arc(a,b), reach(b,a). \\
\ldots \\
reach(c,b). \leftarrow arc(c,d), reach(d,b). \\
\ldots
\end{array}
$$

Reachability — an example

Least model

- $arc(a, b)$, $arc(a, c)$, $arc(d, c)$
- $reach(a, a)$, $reach(b, b)$, $reach(c, c)$, $reach(d, d)$
- $reach(a, b)$, $reach(a, c)$, $reach(d, c)$, $reach(a, c)$

What's computed?

- Assume *reach* is the distinguished “solution” predicate
- $\{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (d, c), (a, c)\}$

Reachability — an example

Least model

- $arc(a, b)$, $arc(a, c)$, $arc(d, c)$
- $reach(a, a)$, $reach(b, b)$, $reach(c, c)$, $reach(d, d)$
- $reach(a, b)$, $reach(a, c)$, $reach(d, c)$, $reach(a, c)$

What's computed?

- Assume *reach* is the distinguished "solution" predicate
- $\{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (d, c), (a, c)\}$

Computing with Horn programs

Possible issues?

- Function symbols necessary!
- List constructor $[\cdot|\cdot]$ used to define higher-order objects
- Terms - basic data structures
- Queries (goals):
  - ?p(t) - is $p(t)$ entailed?
  - ?p(X) - for what ground t, is $p(t)$ entailed?
- Proofs provide answers
- SLD-resolution
- Unification - basic mechanism to manipulate data structures
- Extensive use of recursion
- Leads to Prolog

Example

Manipulating lists: *append* and *reverse*

$$append([\,], X, X).$$
$$append([X|Y], Z, [X|T]) \quad \leftarrow \quad append(Y, Z, T).$$

$$reverse([\,], [\,]).$$
$$reverse([X|Y], Z) \leftarrow \ append(U, [X], Z), reverse(Y, U).$$

- both relations defined recursively
- terms represent complex objects: lists, sets, ...

Example, cont'd

Playing with *reverse*

- Problem: reverse list $[a, b, c]$
  - Ask query $?-reverse([a, b, c], X)$.
  - A proof of the query yields a substitution: $X = [c, b, a]$
  - The substitution constitutes an answer
- Query $?-reverse([a|X], [b, c, d, a])$ returns $X = [d, c, b]$
- Query $?-reverse([a|X], [b, c, d, b])$ returns no substitutions (there is no answer)

Example, cont'd

Observations

- Techniques to search for proofs — the key
- Understanding of the resolution mechanism is important
- It may make a difference which logically equivalent form is used:
  - $reverse([X|Y], Z) \leftarrow append(U, [X], Z), reverse(Y, U).$
  - $reverse([X|Y], Z) \leftarrow reverse(Y, U), append(U, [X], Z).$
  - termination vs. non-termination for query:
    $?-reverse([a|X], [b, c, d, b])$
- Is it truly knowledge representation?
  - is it truly declarative?
  - implementations are not!
- Nonmonotonicity quite restricted

Negation in the body

Why negation?

- Natural linguistic concept
- Facilitates knowledge representation (declarative descriptions and definitions
- Needed for modeling convenience
- Clauses of the form:

$$p(\vec{X}) \leftarrow q_1(\vec{X_1}), \ldots, q_k(\vec{X_k}), not\, r_1(\vec{Y_1}), \ldots, not\, r_l(\vec{Y_l})$$

- Things get more complex!

Semantics of programs with negation

Observations

- Still Herbrand models
- Still restricted to $HB(P)$
- But — usually no least Herbrand model!
- Program

  $$a \leftarrow not\ b$$
  $$b \leftarrow not\ a$$

  has two minimal Herbrand models: $M_1 = \{a\}$ and $M_2 = \{b\}$.
- Identifying a single intended model a major issue

Semantics of programs with negation

Great Logic Programming Schism

- Single intended model approach
  - continue along the lines of Prolog
- Multiple intended model approach
  - branch into answer-set programming

Single intended model approach

“Better” Prolog

- Extensions of Horn logic programming
  - Perfect semantics of stratified programs
  - 3-val well-founded semantics for (arbitrary) programs
- Top-down computing based on unification and resolution
- XSB – David Warren at SUNY Stony Brook
- Will come back to it

Multiple intended models

Answer-set programming

- Semantics assigns to a program not one but many intended models!
  - for instance, all stable or supported models (to be introduced soon)
- How to interpret these semantics?
  - skeptical reasoning: a ground atom is cautiously entailed if it belongs to all intended models
  - intended models represent different possible states of the world, belief sets, solutions to a problem
- Nonmonotonicity shows itself in an essential way
  - as in default logic

Normal logic programming

Preliminary observations and comments

- Logic programs with negation
- Still interested only in Herbrand models
- Thus, enough to consider propositional case
- Supported model semantics
- Stable model semantics
- Connection to propositional logic (Clark's completion, tightness, loop formulas)
- Kripke-Kleene and well-founded semantics
- Strong and uniform equivalence

Normal logic programming — propositional case

Syntax

- Propositional language over a set of atoms At (possibly infinite)
- Clause r

$$a \leftarrow b_1, \ldots, b_m, not\ c_1, \ldots, not\ c_n$$

  - a, b_i, c_j are atoms
  - a is the head of the clause: $hd(r)$
  - literals b_i, $not\ c_j$ form the body of the rule: $bd(r)$
  - $\{b_1, \ldots, b_m\}$ - positive body $bd^+(r)$
  - $\{c_1, \ldots, c_n\}$ - negative body $bd^-(r)$

One-step provability operator

Basic tool in LP
van Emden, Kowalski 1976

- Operator on interpretations (sets of atoms)
- $T_P(I) = \{hd(r) \colon I \models bd(r)\}$
- If P is Horn, T_P is monotone
  - Let $I \subseteq J$
  - Let $bd(r) = b_1, \ldots, b_m$ (no negation!)
  - If $I \models bd(r)$ than $J \models bd(r)$
  - Thus, $T_P(I) \subseteq T_P(J)$
  - Least fixpoint of T_P exists and coincides with the least Herbrand model of P
- In general, not the case (due to negation)
  - $\emptyset \models \mathit{not}\ a$
  - but $\{a\} \not\models \mathit{not}\ a$

Supported-model semantics

Definition and some observations

- $M \subseteq At$ is a supported model of P if $T_P(M) = M$
- Supported models are indeed models
  - let $M \models bd(r)$
  - then $hd(r) \in T_P(M)$
  - and so, $hd(r) \in M$
- Supported models are subsets of $At(P)$ (the Herbrand base of P)
- Thus, they are Herbrand models

Supported models - example

Program $p \leftarrow \textit{not } q$

- One supported model: $M_1 = \{p\}$
- $M_2 = \{q\}$ - not supported (but model)
- p "follows"
- If q included in the program (more exactly: a rule $q \leftarrow$)
  - Just one supported model: $M_1 = \{q\}$.
  - p does not 'follow"
  - nonmonotonicity

Supported models - example

Program $p \leftarrow p$

- Two supported models: $M_1 = \emptyset$ and $M_2 = \{p\}$
- The second one is self-supported (circular justification)
- A problem for KR

Clark's completion

Rules as implications

- $bd^{\wedge}(r)$ the conjunction of all literals in the body of r
  with all not c replaced with $\neg c$
- $cmpl^{\leftarrow}(P) = \{bd^{\wedge}(r) \rightarrow hd(r) \colon r \in P\}$

Clark's completion

Rules as definitions

- Notation: $def_P(a) = \bigvee\{bd^{\wedge}(r) \colon hd(r) = a\}$
- Note: if a not the head of any rule in P, $def_P(a) = \bot$
- $cmpl^{\rightarrow}(P) = \{a \rightarrow def_P(a) \colon a \in At\}$
- $cmpl(P) = cmpl^{\leftarrow}(P) \cup cmpl^{\rightarrow}(P)$
- Note: if $a \notin At(P)$, $cmpl(P) \models \neg a$

Clark's completion

Example

$$
\begin{aligned}
a &\leftarrow b, \mathit{not}\ c \\
a &\leftarrow d \\
b &\leftarrow a
\end{aligned}
$$

- $def(a) = (b \wedge \neg c) \vee d$
- $def(b) = a$
- $def(c) = \bot$
- $cmpl^{\leftarrow} = \{b \wedge \neg c \rightarrow a;\ d \rightarrow a;\ a \rightarrow b\} = \{(b \wedge \neg c) \vee d \rightarrow a;\ a \rightarrow b\}$
- $cmpl^{\leftarrow} = \{def(a) \rightarrow a;\ def(b) \rightarrow b;\ def(c) \rightarrow c\}$
- $cmpl^{\rightarrow} = \{a \rightarrow def(a);\ b \rightarrow def(b);\ c \rightarrow def(c)\}$
- $cmpl = \{a \leftrightarrow def(a);\ b \leftrightarrow def(b);\ c \leftrightarrow def(c)\}\}$
- *cmpl* has two models: $\emptyset$ and $\{a, b\}$

Clark's completion

Connection to supported models

- A set $M \subseteq At$ is a supported model of a program P if and only if M is a model (in a standard sense) of $cmpl(P)$
- Connection to SAT
- Allows us to use SAT solvers to compute supported models of P

Connection to supported models — proof

M — supported model of P: $M = T_P(M)$

- Let $a \in M \Rightarrow \exists r \in P$ st: $hd(r) = a$ and $M \models bd(r)$
- $\Rightarrow M \models bd^\wedge(r)$, $M \models def(a)$ and $M \models a \leftrightarrow def(a)$
- Let $a \notin M \Rightarrow \forall r \in P$ st: $hd(r) = a$, $M \not\models bd(r)$
- $\Rightarrow M \not\models def(a)$ and $M \models a \leftrightarrow def(a)$

Conversely: let $M \models cmpl(P)$

- $\Rightarrow M \models P$ and so, $T_P(M) \subseteq M$
- Let $a \in M \Rightarrow M \models def(a)$
- $\Rightarrow \exists r \in P$ st: $M \models bd^\wedge(r)$
- $\Rightarrow M \models bd(r)$ and $a \in T_P(M) \Rightarrow M \subseteq T_P(M)$
- Thus, $M = T_P(M)$ and M supported

Connection to supported models — proof

M — supported model of P: $M = T_P(M)$

- Let $a \in M \Rightarrow \exists r \in P$ st: $hd(r) = a$ and $M \models bd(r)$
- $\Rightarrow M \models bd^{\wedge}(r)$, $M \models def(a)$ and $M \models a \leftrightarrow def(a)$
- Let $a \notin M \Rightarrow \forall r \in P$ st: $hd(r) = a$, $M \not\models bd(r)$
- $\Rightarrow M \not\models def(a)$ and $M \models a \leftrightarrow def(a)$

Conversely: let $M \models cmpl(P)$

- $\Rightarrow M \models P$ and so, $T_P(M) \subseteq M$
- Let $a \in M \Rightarrow M \models def(a)$
- $\Rightarrow \exists r \in P$ st: $M \models bd^{\wedge}(r)$
- $\Rightarrow M \models bd(r)$ and $a \in T_P(M) \Rightarrow M \subseteq T_P(M)$
- Thus, $M = T_P(M)$ and M supported

Stable model semantics

Supported models of interest, but ...

- Some supported models based on circular arguments
  - $p \leftarrow p$
  - $\{p\}$ is supported model (circular argument)
- Some more stringent bases for selecting intended models needed

Stable model semantics

Gelfond-Lifschitz reduct

- P — logic program
- M — set of atoms
- Reduct P^M
  - for each $a \in M$ remove rules with *not a* in the body
  - remove literals *not a* from all other rules

Stable model semantics

Definition through reduct

- Reduct P^M is a Horn program
- It has the least model $LM(P^M)$
- M is a stable model of P if

$$M = LM(P^M)$$

Stable model semantics

And now through Gelfond-Lifschitz operator

- $GL_P(M) = LM(P^M)$
- M is a stable model if and only if

$$M = GL_P(M)$$

- GL_P is antimonotone
- For $M \subseteq N$:

$$GL_P(N) \subseteq GL_P(M)$$

Stable models — examples

Multiple stable models

$$\begin{aligned} p &\leftarrow q, not\, s \\ r &\leftarrow p, not\, q, not\, s \\ s &\leftarrow not\, q \\ q &\leftarrow not\, s \end{aligned}$$

- Two stable models: $M_1 = \{p, q\}$ and $M_2 = \{s\}$

No stable models

$$p \leftarrow not\, p$$

- No stable models!!

Stable models — examples

Multiple stable models

$$
\begin{aligned}
p &\leftarrow q, not\ s \\
r &\leftarrow p, not\ q, not\ s \\
s &\leftarrow not\ q \\
q &\leftarrow not\ s
\end{aligned}
$$

- Two stable models: $M_1 = \{p, q\}$ and $M_2 = \{s\}$

No stable models

$$
p \leftarrow not\ p
$$

- No stable models!!

Stable models — properties

Stable models are models!

- Let M be a stable model
- M is a model of all rules that are removed from the program when forming the reduct
- M is a model of every rule that contributes to the reduct
- Indeed, each such rule is subsumed by a rule in the reduct and M satisfies all rules in the reduct

Stable models — properties

Stable models are minimal models!

- Let N be a stable model and M a model s.t. $M \subseteq N$
- Then
$$N = GL_P(N) \subseteq GL_P(M) \subseteq M$$
- Thus, $N \subseteq M$ and so $N = M$
- The minimality of N follows
- Stable models form an antichain!

Stable models — properties

Lemma: If M model of P, $GL_P(M) \subseteq M$

- Since M model of P, then M is a model of P^M
- $a \leftarrow b_1, \ldots, b_m$ - a rule in reduct
- $a \leftarrow b_1, \ldots, b_m, not\ c_1, \ldots, not\ c_n$ - the original rule in P
- M satisfies the latter, and it satisfies every $not\ c_i$ (as $c_i \notin M$)
- Thus, M satisfies the reduct rule

Stable models — properties

Connection to supported models

- If M is a stable model of P then it is a supported model of P
- Let M be a stable model of P
- Then M model of P and so, $T_P(M) \subseteq M$
- $r = \quad a \leftarrow b_1, \ldots, b_m, not\ c_1, \ldots, not\ c_n$ - a rule in P such that $M \models bd(r)$
- Then $r' \quad = a \leftarrow b_1, \ldots, b_m$ belongs to the reduct P^M
- And $M \models bd(r')$
- Since M is a model of P^M, $a \in M$
- Hence, $T_P(M) \subseteq M$ and so, $M = T_P(M)$
- That is, M is supported!!

Fages Lemma

Fages 1994

But ...

- The converse not true, in general (as it should not be)

Counterexample

- $p \leftarrow p$
- $\{p\}$ is supported but not stable
- Positive dependency of p on itself is a problem

Fages Lemma

But ...

- The converse not true, in general (as it should not be)

Counterexample

- $p \leftarrow p$
- $\{p\}$ is supported but not stable
- Positive dependency of p on itself is a problem

Fages Lemma

Positive dependency graph $G^+(P)$

- Atoms of P are vertices
- (a, b) is an edge in $G^+(P)$ if for some $r \in P$: $hd(r) = a$, $b \in bd^+(r)$

Tight programs

- P is tight if $G^+(P)$ is acyclic
- Alternatively, if there is a labeling of atoms with non-negative integers $(a \mapsto \lambda(a))$ s.t.
- for every rule $r \in P$

$$\lambda(hd(r)) > \max\{\lambda(b) : b \in bd^+(r)\}$$

- Connection to topological ordering of positive dependency graphs

Fages Lemma

Positive dependency graph $G^+(P)$

- Atoms of P are vertices
- (a, b) is an edge in $G^+(P)$ if for some $r \in P$: $hd(r) = a$, $b \in bd^+(r)$

Tight programs

- P is tight if $G^+(P)$ is acyclic
- Alternatively, if there is a labeling of atoms with non-negative integers ($a \mapsto \lambda(a)$) s.t.
- for every rule $r \in P$

$$\lambda(hd(r)) > \max\{\lambda(b) : b \in bd^+(r)\}$$

- Connection to topological ordering of positive dependency graphs

Fages Lemma

The statement — finally

- If P is tight then every supported model is stable
- Intuitively, circular support not possible

Fages Lemma — example

Program P

$$\begin{array}{lcl} p & \leftarrow & q, not\ s \\ r & \leftarrow & p, not\ q, not\ s \\ s & \leftarrow & not\ q \\ q & \leftarrow & not\ s \end{array}$$

Graph $G^+(P)$

P is tight

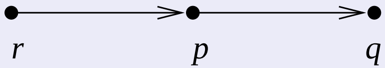
- $\{p, q\}$ and $\{s\}$ are supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Thus, they are stable (which we verified directly earlier)

Fages Lemma — example

Program P

$$\begin{array}{lcl} p & \leftarrow & q, not\, s \\ r & \leftarrow & p, not\, q, not\, s \\ s & \leftarrow & not\, q \\ q & \leftarrow & not\, s \end{array}$$

Graph $G^+(P)$



P is tight

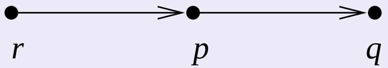
- $\{p, q\}$ and $\{s\}$ are supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Thus, they are stable (which we verified directly earlier)

Fages Lemma — example

Program P

$$
\begin{aligned}
p &\leftarrow q, not\, s\\
r &\leftarrow p, not\, q, not\, s\\
s &\leftarrow not\, q\\
q &\leftarrow not\, s
\end{aligned}
$$

Graph $G^+(P)$



P is tight

- $\{p, q\}$ and $\{s\}$ are supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Thus, they are stable (which we verified directly earlier)

Fages Lemma

Proof

- Let P be tight and M be its supported model
- Then M is a model of P^M
- Let N be a model of P^M
- Claim: for every k, if $a \in M$ and $\lambda(a) < k$, then $a \in N$
- Holds for $k = 0$ (trivially)
- Let $a \in M$ and $\lambda(a) = k$
- Since M supported, there is $r \in P$ such that $a = hd(r)$ and $M \models bd(r)$
- In particular, $bd^+(r) \subseteq M$ and so, $bd^+(r) \subseteq N$ (by I.H.)
- Since $M \models bd(r)$, M contributes to the reduct
- Since N is a model of P^M, $a \in N$
- It follows that $M = LM(P^M)$

A generalization

Erdem and Lifschitz, 2000

Relativized tightness

- Let $X \subseteq At(P)$
- P is tight on X if the program consisting of rules r such that $bd^{+}(r) \subseteq X$ is tight

Generalization

- If P is tight on X and M is a supported model of P such that $M \subseteq X$, then M is stable

A generalization

Erdem and Lifschitz, 2000

Relativized tightness

- Let $X \subseteq At(P)$
- P is tight on X if the program consisting of rules r such that $bd^+(r) \subseteq X$ is tight

Generalization

- If P is tight on X and M is a supported model of P such that $M \subseteq X$, then M is stable

Generalized Fages Lemma — example

Program P

$$
\begin{array}{lcl}
p & \leftarrow & q, \mathit{not}\ s \\
r & \leftarrow & p, \mathit{not}\ q, \mathit{not}\ s \\
s & \leftarrow & \mathit{not}\ q \\
q & \leftarrow & \mathit{not}\ s \\
p & \leftarrow & r
\end{array}
$$

Graph $G^+(P)$

P is not tight

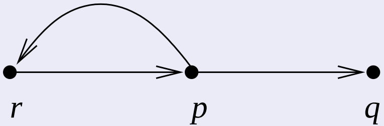
- $\{p, q\}$ and $\{s\}$ are still supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Since P is tight on each of them, they are stable

Generalized Fages Lemma — example

Program P

$$
\begin{aligned}
p &\leftarrow q, not\ s \\
r &\leftarrow p, not\ q, not\ s \\
s &\leftarrow not\ q \\
q &\leftarrow not\ s \\
p &\leftarrow r
\end{aligned}
$$

Graph $G^+(P)$



P is not tight

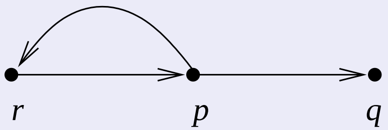
- $\{p, q\}$ and $\{s\}$ are still supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Since P is tight on each of them, they are stable

Generalized Fages Lemma — example

Program P

$$
\begin{array}{lll}
p & \leftarrow & q, \mathit{not}\, s \\
r & \leftarrow & p, \mathit{not}\, q, \mathit{not}\, s \\
s & \leftarrow & \mathit{not}\, q \\
q & \leftarrow & \mathit{not}\, s \\
p & \leftarrow & r
\end{array}
$$

Graph $G^+(P)$



P is not tight

- $\{p, q\}$ and $\{s\}$ are still supported models of P
  - $T_P(\{p, q\}) = \{p, q\}$
  - $T_P(\{s\}) = \{s\}$
- Since P is tight on each of them, they are stable

External support formula for $Y \subseteq At(P)$

- For a rule r:
- $bd^{\wedge}(r)$ the conjunction of all literals in the body of r

  with all not c replaced with $\neg c$
- For $Y \neq \emptyset$:
  - $ES_P(Y)$ the disjunction of $bd^{\wedge}(r)$ for all $r \in P$ st:
  - $hd(r) \in Y$
  - $bd^{+}(r) \cap Y = \emptyset$
- For finite programs: well-formed formulas
- Hence, will assume finiteness

Observations

- $ES_P(\{a\}) = def_P(a)$

  cf. Clark's completion

External support formula for $Y \subseteq At(P)$

- For a rule r:
- $bd^{\wedge}(r)$ the conjunction of all literals in the body of r

  with all not c replaced with $\neg c$
- For $Y \neq \emptyset$:
  - $ES_P(Y)$ the disjunction of $bd^{\wedge}(r)$ for all $r \in P$ st:
  - $hd(r) \in Y$
  - $bd^{+}(r) \cap Y = \emptyset$
- For finite programs: well-formed formulas
- Hence, will assume finiteness

Observations

- $ES_P(\{a\}) = def_P(a)$

  cf. Clark's completion

A characterization of stable models

for finite programs, the following conditions are equivalent

- X is a stable model of P
- X is a model of $cmpl^{\leftarrow}(P) \cup \{Y^{\wedge} \rightarrow ES_P(Y) \colon Y \subseteq At(P),\ Y \neq \emptyset\}$
- X is a model of $cmpl^{\leftarrow}(P) \cup \{Y^{\vee} \rightarrow ES_P(Y) \colon Y \subseteq At(P),\ Y \neq \emptyset\}$
- OK to replace $cmpl^{\leftarrow}(P)$ with $cmpl(P)$
  - $cmpl^{\rightarrow}(P) \subseteq \{Y^{\wedge} \rightarrow ES_P(Y) \colon Y \subseteq At(P)\}$
  - $cmpl^{\rightarrow}(P) \subseteq \{Y^{\vee} \rightarrow ES_P(Y) \colon Y \subseteq At(P)\}$

Loops

Definition

- A loop is a non-empty set $Y \subseteq At(P)$ that induces in $G^+(P)$ a strongly connected subgraph
- In particular, all singleton sets are loops

Loops — example

Program P

$$
\begin{aligned}
p &\leftarrow q, not\ r \\
q &\leftarrow p \\
r &\leftarrow not\ p
\end{aligned}
$$

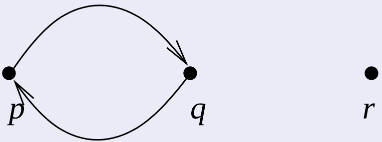
Graph $G^+(P)$

- $\{p\}$, $\{q\}$, $\{r\}$, $\{p, q\}$ are loops
- $\{p, q, r\}$ is not!

Program P

$$
\begin{array}{lcl}
p & \leftarrow & q, \mathit{not}\ r \\
q & \leftarrow & p \\
r & \leftarrow & \mathit{not}\ p
\end{array}
$$

Graph $G^+(P)$



- $\{p\}$, $\{q\}$, $\{r\}$, $\{p,q\}$ are loops
- $\{p,q,r\}$ is not!

Loop Theorem

For finite programs, the following conditions are equivalent

- X is a stable model of P
- X is a model of $cmpl^{\leftarrow}(P) \cup \{Y^{\wedge} \rightarrow ES_P(Y) : Y$ – a loop$\}$
- X is a model of $cmpl^{\leftarrow}(P) \cup \{Y^{\vee} \rightarrow ES_P(Y) : Y$ – a loop$\}$
- OK to replace $cmpl^{\leftarrow}(P)$ with $cmpl(P)$
  - Singleton sets are loops!

Loop Theorem

Let X be a stable model of P

- $\Rightarrow \quad X \models P \quad \Rightarrow \quad X \models P^X$
- $X \models P \quad \Rightarrow \quad X \models cmpl^{\leftarrow}(P)$
- Let Y be a loop st: $X \models Y^{\wedge} \quad \Rightarrow \quad X \cap Y \neq \emptyset$
- Let $a \in Y$ be the "first" element of Y in X
  recall that $X = LM(P^X)$
- $\Rightarrow \quad \exists r \in P$ st: $\quad a = hd(r), \quad bd^+(r) \cap Y = \emptyset$
- $\Rightarrow \quad bd^{\wedge}(r)$ is a disjunct of $ES_P(Y)$
- Also: $\quad bd^+(r) \subseteq X \quad$ and $\quad bd^-(r) \cap X = \emptyset \Rightarrow \quad X \models bd^{\wedge}(r)$
- $\Rightarrow \quad X \models ES_P(Y) \quad \Rightarrow \quad X \models Y^{\wedge} \rightarrow ES_P(Y)$
- No difference if $Y^{\wedge}$ replaced with $Y^{\vee}$

Loop Theorem

Let $X \models cmpl^{\leftarrow}(P) \cup \{Y^{\wedge} \rightarrow ES_P(Y) \colon Y \text{ – a loop}\}$

- $\Rightarrow \quad X \models P \quad \Rightarrow \quad X \models P^X$
- Let $X' = LM(P^X) \quad \Rightarrow \quad X' \subseteq X$
- Let $X \setminus X' \neq \emptyset$
- Consider subgraph H of $G^{(}P)$ induced by $X \setminus X'$
- Let Y be a “terminal” strongly connected component of H

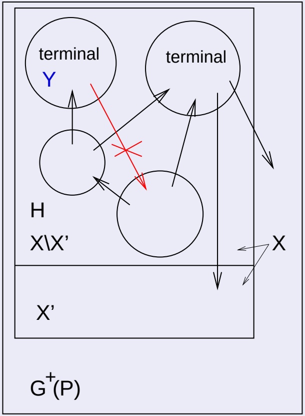
  no edge in H starts in Y and ends outside of Y

Loop Theorem

Let $X \models cmpl^{\leftarrow}(P) \cup \{Y^{\wedge} \rightarrow ES_P(Y) :\ Y$ – a loop$\}$

- $\Rightarrow \quad X \models P \quad \Rightarrow \quad X \models P^X$
- Let $X' = LM(P^X) \quad \Rightarrow \quad X' \subseteq X$
- Let $X \setminus X' \neq \emptyset$
- Consider subgraph H of $G^{(}P)$ induced by $X \setminus X'$
- Let Y be a "terminal" strongly connected component of H

  no edge in H starts in Y and ends outside of Y



Loop Theorem

Proof, cont'd

- $X \models Y^{\wedge} \rightarrow ES_P(Y)$ (also: $X \models Y^{\vee} \rightarrow ES_P(Y)$)
- Since $Y \subseteq X$: $\Rightarrow$ $X \models ES_P(Y)$
- $\Rightarrow$ $\exists r \in P$ st: $hd(r) \in Y$, $bd^+(r) \cap Y = \emptyset$, $X \models bd^{\wedge}(r)$
- $\Rightarrow$ $bd^+(r) \subseteq X$ and $r^X \in P^X$
- Since Y terminal and $bd^+(r) \cap Y = \emptyset$: $\Rightarrow$ $bd^+(r) \subseteq X'$
  - if $a \in bd^+(r)$: $a \in X$
  - $(hd(r), a)$ edge in $G^+(P)$
  - if $a \in X \setminus X'$: $(hd(r), a)$ edge in H
  - $\Rightarrow$ $a \in Y$, contradiction
  - $\Rightarrow$ $a \in X'$
- Since $X' \models P^X$: $\Rightarrow$ $X' \models r^X$
- $\Rightarrow$ $hd(r) \in X'$
- Since $X' \cap Y = \emptyset$: $\Rightarrow$ contradiction

Program inconsistency

Some programs have no stable nor supported models

- Sufficient conditions for the existence of stable models
- 4-val supported and stable models

Sufficient conditions

General dependency graph $G(P)$

- Atoms of P are vertices
- (a, b) is an edge in P if for some $r \in P$: $hd(r) = a, b \in bd(r)$
- If $b \in bd^+(r)$ — edge is positive
- If $b \in bd^-(r)$ — edge is negative

A propositional program P is

- Call-consistent: if $G(P)$ has no odd cycles (cycles with an odd number of negative edges)
- Stratified: if $G(P)$ has no paths with infinitely many negative edges
  - in particular, no cycles with a negative edge (for finite programs both conditions are equivalent)

Sufficient conditions

General dependency graph $G(P)$

- Atoms of P are vertices
- (a, b) is an edge in P if for some $r \in P$: $\quad hd(r) = a$, $b \in bd(r)$
- If $b \in bd^{+}(r)$ — edge is positive
- If $b \in bd^{-}(r)$ — edge is negative

A propositional program P is

- Call-consistent: if $G(P)$ has no odd cycles (cycles with an odd number of negative edges)
- Stratified: if $G(P)$ has no paths with infinitely many negative edges
  - in particular, no cycles with a negative edge (for finite programs both conditions are equivalent)

Sufficient conditions

Results

- If a finite logic program is call-consistent, it has a stable model
- If a program is stratified it has a unique stable model

Stratification through splitting

Splitting

- Let P and Q be programs such that P does not contain any of the head atoms of Q

  "Q above P"
- A set M is a stable model of $P \cup Q$ iff there is a stable model N of P such that M is a stable model of $Q \cup N$

Splitting Theorem

Proof: ($\Rightarrow$) Let $M \in StM(P \cup Q)$

- $N := M \cap At(P)$
- $P^N = P^M$ (as $(M \setminus N) \cap At(P) = \emptyset$)
- $M \models P \quad \Rightarrow \quad M \models P^M \quad \Rightarrow \quad M \models P^N$
- $\Rightarrow \quad N \models P^N$ (as $(M \setminus N) \cap At(P) = \emptyset$)
- Let $N' \subseteq N$ be st: $\quad N' \models P^N$
- $\Rightarrow \quad N' \models P^M \quad \Rightarrow \quad N' \cup (M \setminus N) \models P^M$
- Let $r \in Q^M$ be st: $\quad N' \cup (M \setminus N) \models bd(r)$
- $\Rightarrow \quad M \models bd(r) \quad \Rightarrow \quad M \models hd(r)$ (as $M \models Q$ and so, $M \models Q^M$)
- $\Rightarrow \quad hd(r) \in M \quad \Rightarrow \quad hd(r) \in M \setminus N \quad \Rightarrow \quad hd(r) \in N' \cup (M \setminus N)$
- $\Rightarrow \quad N' \cup (M \setminus N) \models Q^M \quad \Rightarrow \quad N' \cup (M \setminus N) \models (P \cup Q)^M$
- $\Rightarrow \quad N' \cup (M \setminus N) = M \quad \Rightarrow \quad N' = N \quad \Rightarrow \quad N = LM(P^N)$
- $\Rightarrow \quad N \in StM(P)$

Splitting Theorem

Next, we show that $M \in StM(Q \cup N)$

- Recall: $N = M \cap At(P) \quad \Rightarrow \quad N \subseteq M$
- Also: $M \models Q \quad \Rightarrow \quad M \models Q^M \cup N = (Q \cup N)^M$
- Let $M' \subseteq M$ be st: $M' \models (Q \cup N)^M$
- $\Rightarrow \quad N \subseteq M' \quad M' \models Q^M$
- Recall: $N \models P^N$ and so $N \models P^M \quad$ (as $P^M = P^N$)
- $\Rightarrow \quad M' \models P^M \quad \Rightarrow \quad M' \models (P \cup Q)^M$
- Recall: $M = LM((P \cup Q)^M) \quad \Rightarrow \quad M = M'$
- $\Rightarrow \quad M = LM((P \cup Q)^M) \quad \Rightarrow \quad M \in StM(Q \cup N)$

Splitting Theorem

Conversely: $M \in StM(Q \cup N)$ and $N \in StM(P)$

- $\Rightarrow \quad M \models Q, \quad N \subseteq M, \quad M \subseteq hd(Q) \cup N$
- $\Rightarrow \quad M \cap At(P) = N \quad \Rightarrow \quad M \models P$
- $\Rightarrow \quad M \models P \cup Q \quad \Rightarrow \quad M \models (P \cup Q)^M$
- Let $M' \subseteq M$ be st: $\quad M' \models (P \cup Q)^M$
- $N' := M' \cap At(P)$
- $\Rightarrow \quad M' \models P^M \quad \Rightarrow \quad N' \models P^M \quad \Rightarrow \quad N' \models P^N$
- $\Rightarrow \quad N' = N \quad \Rightarrow \quad N \subseteq M' \quad \Rightarrow \quad M' \models Q^M \cup N = (Q \cup N)^M$
- $\Rightarrow \quad M' = M \quad \Rightarrow \quad M = LM((Q \cup N)^M \quad \Rightarrow \quad M \in StM(P \cup Q)$

Stratification

Equivalent definition in the finite case

- P stratified if
  - $hd(P) \cap bd^{-}(P) = \emptyset$, or
  - $P = P_1 \cup P_2$, where P_2 stratified, $hd(P_1) \cap (bd^{-}(P_1) \cup At(P_2)) = \emptyset$

Finite stratified programs have a unique stable model

- Induction
- Basis: exident
- Inductive step: P_2 has a unique stable model, say N
- Clearly, $P_1 \cup N$ has a unique stable model, too
- Apply splitting theorem

Equivalence — logics behind nonmonotonic logics

What do I mean?

- Logic allows us to manipulate theories
- Tautologies can be added or removed without changing the meaning
- Consequences of formulas in theories can be added or removed without changing the meaning
  - $\{p, p \rightarrow q\}$ the same as $\{p, p \rightarrow q, q\}$
  - one can always be replaced with another (within any larger context)
- Equivalence for replacement — logical equivalence necessary and sufficient
- Is there a logic which captures such manipulation with theories in nonmonotonic systems?

Is it important?

Query optimization

- Compute answers to a query Q (program) from a knowledge base KB (another program)
  reason from $Q \cup KB$
- Rewrite Q into an equivalent query Q', which can be processed more efficiently
  reasoning from $Q' \cup KB$ easier
- When are two queries equivalent?
  - If $Q \cup KB$ and $Q' \cup KB$ have the same meaning
    not quite what we want — knowledge-base dependent
  - If $Q \cup KB$ and $Q' \cup KB$ have the same meaning for every knowledge base KB
    better — knowledge-base independent

Towards modular logic programming

Equivalence of programs

- P and Q are equivalent if they have the same models

Nonmonotonic equivalence of programs

- P and Q are stable-equivalent if they have the same stable models

Towards modular logic programming

Equivalence of programs

- P and Q are equivalent if they have the same models

Nonmonotonic equivalence of programs

- P and Q are stable-equivalent if they have the same stable models

Equivalence for replacement

- Equivalence for replacement — for every program R, programs $P \cup R$ and $Q \cup R$ have the same stable models
- Commonly known as strong equivalence

  Lifschitz, Pearce, Valverde 2001; Lin 2002; Turner 2003; Eiter, Fink 2003; Eiter, Fink, Tompits, Woltran, 2005; T_ 2006; Woltran 2008
- Different than equivalence
  - $\{p \leftarrow not\ q\}$ and $\{q \leftarrow not\ p\}$
  - The same models but different meaning
- Different than stable-equivalence
  - $P = \{p\}$ and $Q = \{p \leftarrow not\ q\}$
  - The same stable models; $\{p\}$ is the only stable model in each case
  - But, $P \cup \{q\}$ and $Q \cup \{q\}$ have different stable models! ($\{p, q\}$ and $\{q\}$, respectively)

When are two programs strongly equivalent?

Se-model characterization

- A pair (X, Y) of sets of atoms is an *se-model* of a program P if
  - $X \subseteq Y$
  - $Y \models P$
  - $X \models P^Y$
- $SE(P)$ set of se-models of P
- Logic programs P and Q are strongly equivalent iff they have the same se-models ($SE(P) = SE(Q)$)
  - A similar concept characterizes strong equivalence of default theories

    Turner 2003

When are two programs strongly equivalent?

Lemma 1: $SE(P) = SE(Q) \quad \Rightarrow \quad StM(P) = StM(Q)$

- $Y \in StM(P) \quad \Rightarrow \quad Y \models P$ and $Y \models P^Y$
- $\Rightarrow \quad (Y, Y) \in SE(P) \quad \Rightarrow \quad (Y, Y) \in SE(Q)$
- $\Rightarrow \quad Y \models Q^Y$
- If $Z \subseteq Y$ and $Z \models Q^Y \quad \Rightarrow \quad (Z, Y) \in SE(Q)$
- $\Rightarrow \quad (Z, Y) \in SE(P)$
- $\Rightarrow \quad Z \models P^Y \quad \Rightarrow \quad Z = Y$ (as $Y = LM(P^Y)$)
- $\Rightarrow \quad Y = LM(Q^Y) \quad \Rightarrow \quad Y \in StM(Q)$

When are two programs strongly equivalent?

Lemma 2: $SE(P \cup R) = SE(P) \cap SE(R)$

- $(X, Y) \in SE(P \cup R)$ iff
- $X \subseteq Y$ and $Y \models P \cup R$ and $X \models (P \cup R)^Y = P^Y \cup R^Y$ iff
- $X \subseteq Y$ and ($Y \models P$ and $Y \models R$) and ($X \models P^Y$ and $X \models R^Y$) iff
- ($X \subseteq Y$, $Y \models P$, $X \models P^Y$), and
  ($X \subseteq Y$, $Y \models R$, $X \models R^Y$) iff
- $(X, Y) \in SE(P)$ and $(X, Y) \in SE(R)$ iff
- $(X, Y) \in SE(P) \cap SE(R)$

When are two programs strongly equivalent?

$SE(P) = SE(Q) \quad \Rightarrow \quad P$ and Q are strongly equivalent

- By Lemma 2, for every R:
  $SE(P \cup R) = SE(P) \cap SE(R) = SE(Q) \cap SE(R) = SE(Q \cup R)$
- By Lemma 1, $StM(P \cup R) = StM(Q \cup R)$

P and Q are strongly equivalent $\quad \Rightarrow \quad SE(P) = SE(Q)$

- Let $(X, Y) \in SE(P) \setminus SE(Q)$: $\quad (X, Y) \in SE(P)$ and $(X, Y) \notin SE(Q)$
- $\Rightarrow \quad Y \models P^Y \quad \Rightarrow \quad Y = LM(P^Y \cup Y)$
- Since $P^Y \cup Y = (P \cup Y)^Y$, $\quad Y = LM((P \cup Y)^Y) \quad \Rightarrow$ $Y \in StM(P \cup Y)$
- $\Rightarrow \quad Y \in StM(Q \cup Y) \quad \Rightarrow \quad Y \models Q$
- $\Rightarrow \quad X \not\models Q^Y$

When are two programs strongly equivalent?

$SE(P) = SE(Q) \quad \Rightarrow \quad P$ and Q are strongly equivalent

- By Lemma 2, for every R: $SE(P \cup R) = SE(P) \cap SE(R) = SE(Q) \cap SE(R) = SE(Q \cup R)$
- By Lemma 1, $StM(P \cup R) = StM(Q \cup R)$

P and Q are strongly equivalent $\quad \Rightarrow \quad SE(P) = SE(Q)$

- Let $(X, Y) \in SE(P) \setminus SE(Q)$: $\quad (X, Y) \in SE(P)$ and $(X, Y) \notin SE(Q)$
- $\Rightarrow \quad Y \models P^Y \quad \Rightarrow \quad Y = LM(P^Y \cup Y)$
- Since $P^Y \cup Y = (P \cup Y)^Y, \quad Y = LM((P \cup Y)^Y) \quad \Rightarrow$ $Y \in StM(P \cup Y)$
- $\Rightarrow \quad Y \in StM(Q \cup Y) \quad \Rightarrow \quad Y \models Q$
- $\Rightarrow \quad X \not\models Q^Y$

When are two programs strongly equivalent?

$(X, Y) \in SE(P)$, $(X, Y) \notin SE(Q)$, $Y \models Q$, $X \not\models Q^Y$

- Define $R = X \cup \{y \leftarrow y' \mid y, y' \in Y \setminus X\}$
- $\Rightarrow \quad Y \models Q \cup R$ and $Y \models (Q \cup R)^Y$
- Let $Z \subseteq Y$ st: $\quad Z \models (Q \cup R)^Y \quad \Rightarrow \quad Z \models Q^Y \cup R$
- $\Rightarrow \quad Z \models Q^Y \quad \Rightarrow \quad X \neq Z$
- Since $Z \models R$, $\quad X \subseteq Z \quad \Rightarrow \quad \exists y \in Y \setminus X$ st: $\quad y \in Z$
- Since $Z \models R$, $\quad Y \setminus X \subseteq Z$
- $\Rightarrow \quad Y \subseteq Z \quad \Rightarrow \quad Z = Y$
- $\Rightarrow \quad Y \in StM(Q \cup R) \quad \Rightarrow \quad Y \in StM(P \cup R)$
- $\Rightarrow \quad Y = LM((P \cup R)^Y)$
- Since $X \models P^Y \cup R = (P \cup R)^Y$, $\quad X = Y$
- $\Rightarrow \quad Y \not\models Q^Y \quad \Rightarrow \quad Y \not\models Q$, a contradiction

An interesting variant

Uniform equivalence

- Programs P and Q are uniformly equivalent if for every set D of facts (rules with empty body) $P \cup D$ and $Q \cup D$ have the same stable models
- Relevant for DB query optimization
- Different than other types of equivalence discussed here

When are two programs uniformly equivalent?

Se-model characterization

- Programs P and Q are uniformly equivalent iff
  - for every $Y \subseteq At$, Y is a model of P if and only if Y is a model of Q
  - for every $(X, Y) \in SE(P)$ such that $X \subset Y$, there is $U \subseteq At$ such that $X \subseteq U \subset Y$ and $(U, Y) \in SE(Q)$
  - for every $(X, Y) \in SE(Q)$ such that $X \subset Y$, there is $U \subseteq At$ such that $X \subseteq U \subset Y$ and $(U, Y) \in SE(P)$

When are two programs uniformly equivalent?

Ue-model characterization

- A pair (X, Y) of sets of atoms is a *ue-model* of a program P if it is an se-model of P and
- For every se-model (X', Y) such that $X \subseteq X'$, $X' = X$ or $X' = Y$
- Finite logic programs P and Q are uniformly equivalent iff they have the same ue-models

  Eiter and Fink, 2003

General logic programs

Formulas

- Base: atoms and the symbol $\bot$ (“false”)
- Connectives $\land$, $\lor$ and $\rightarrow$
- Shortcuts
  - $\neg F ::= F \rightarrow \bot$
  - $\top ::= \bot \rightarrow \bot$
  - $F \leftrightarrow G ::= (F \rightarrow G) \land (G \rightarrow F)$

General logic programs

Positive and negative occurrences of atoms in formulas

- An occurrence of a in F is positive, if the # of implications with this occurrence of a in antecedent is even
- Otherwise, it is negative
- An occurrence of a in F is strictly positive if no implication contains this occurrence of a in the antecedent
  - $\neg F$ (that is, $F \to \bot$) has no strictly positive occurrences of any atom.
- A head atom (of a formula)   an atom with at least one strictly positive occurrence
- In $(\neg p \to q) \to (p \vee \neg q)$:
  - the first occurrence of p is negative
  - the second occurrence of p is strictly positive
  - both occurrences of q are negative

Stable-model semantics

Reduct of a formula F with respect to a set X of atoms

- The formula F^X obtained by replacing in F each maximal subformula of F that is not satisfied by X with $\bot$

Example: $F = (\neg p \rightarrow q) \wedge (\neg q \rightarrow p)$ and $X = \{p\}$

- $\neg p = p \rightarrow \bot$, and $X \not\models \neg p \rightarrow q$
- Thus: $\neg p$ is a maximal subformula not satisfied by X
- $\neg q = q \rightarrow \bot$, $X \not\models q$, $X \models \neg q$
- Thus, q is a maximal subformula not satisfied by X
- Thus: $F^X = (\bot \rightarrow q) \wedge ((\bot \rightarrow \bot) \rightarrow p)$
- Classically equivalent to p

Stable-model semantics

Reduct of a formula F with respect to a set X of atoms

- The formula F^X obtained by replacing in F each maximal subformula of F that is not satisfied by X with $\bot$

Example: $F = (\neg p \to q) \wedge (\neg q \to p)$ and $X = \{p\}$

- $\neg p = p \to \bot$, and $X \models \neg p \to q$
- Thus: $\neg p$ is a maximal subformula not satisfied by X
- $\neg q = q \to \bot$, $X \not\models q$, $X \models \neg q$
- Thus, q is a maximal subformula not satisfied by X
- Thus: $F^X = (\bot \to q) \wedge ((\bot \to \bot) \to p)$
- Classically equivalent to p

Stable-model semantics

To facilitate computation of the reduct

- $\perp^X = \perp$
- For a an atom, if $a \in X$, $a^X = a$; otherwise, $a^X = \perp$
- If $X \models F \circ G$, $(F \circ G)^X = F^X \circ G^X$; otherwise, $(F \circ G)^X = \perp$ ($\circ$ stands for any of $\wedge$, $\vee$, $\rightarrow$)
- If $X \models F$, $(\neg F)^X = \perp$; otherwise, $(\neg F)^X = (F \rightarrow \perp)^X = F^X \rightarrow \perp^X = \perp \rightarrow \perp = \top$

Stable-model semantics

Definition

- A set X of atoms is a *stable model* of a formula F if X is a minimal model of F

Example: $F = (\neg p \to q) \wedge (\neg q \to p)$, $X = \{p\}$

- $F^X = (\bot \to q) \wedge ((\bot \to \bot) \to p)$ (which is equivalent to p)
- X is a minimal model of F^X, so a stable model

Example: $F = (\neg p \to q) \wedge (\neg q \to p)$, $X = \{p, q\}$

- $F^X = (\bot \to q) \wedge (\bot \to p)$ (which is equivalent to $\top$)
- X is not a minimal model of F^X, so not a stable model

Stable-model semantics

Definition

- A set X of atoms is a *stable model* of a formula F if X is a minimal model of F

Example: $F = (\neg p \to q) \wedge (\neg q \to p)$, $X = \{p\}$

- $F^X = (\bot \to q) \wedge ((\bot \to \bot) \to p)$ (which is equivalent to p)
- X is a minimal model of F^X, so a stable model

Example: $F = (\neg p \to q) \wedge (\neg q \to p)$, $X = \{p, q\}$

- $F^X = (\bot \to q) \wedge (\bot \to p)$ (which is equivalent to $\top$)
- X is not a minimal model of F^X, so not a stable model

Stable-model semantics

Definition

- A set X of atoms is a *stable model* of a formula F if X is a minimal model of F

Example: $F = (\neg p \rightarrow q) \wedge (\neg q \rightarrow p), X = \{p\}$

- $F^X = (\bot \rightarrow q) \wedge ((\bot \rightarrow \bot) \rightarrow p)$ (which is equivalent to p)
- X is a minimal model of F^X, so a stable model

Example: $F = (\neg p \rightarrow q) \wedge (\neg q \rightarrow p), X = \{p, q\}$

- $F^X = (\bot \rightarrow q) \wedge (\bot \rightarrow p)$ (which is equivalent to $\top$)
- X is not a minimal model of F^X, so not a stable model

Stable-model semantics

Properties

- If X is a stable model of a formula F then X consists of head atoms of F
- A least model of a Horn formula (conjunction of definite Horn clauses given as implications) is a unique stable model of the theory
- A set X is a stable model of a formula $F \wedge \neg G$ if and only if X is a stable model of F and $X \models \neg G$

Stable-model semantics

Strong equivalence

- Formulas F and F' are strongly equivalent if for every formula G, $F \wedge G$ and $F' \wedge G$ have the same stable models
- (X, Y) is an *se-model* of F if $Y \subseteq At$, $X \subseteq Y$, $Y \models F$ and $X \models F^Y$.
- The following conditions are equivalent:
  - Formulas F and G are strongly equivalent
  - For every set X of atoms, F^X and G^X are equivalent in classical logic
  - F and G have the same se-models
  - F and G are equivalent in the logic here-and-there (details later)

Stable-model semantics

Splitting

- Let F and G be formulas such that F does not contain any of the head atoms of G
- A set X is a stable model of $F \wedge G$ iff there is a stable model Y of F such that X is a stable model of $G \wedge \bigwedge Y$

Multivalued semantics

2-input one-step operator Φ_P

- Given two interpretations I and J

$$\Phi_P(I, J) = \{hd(r) \colon r \in P,\ bd^+(r) \subseteq I,\ bd^-(r) \cap J = \emptyset\}$$

- $\Phi_P(\cdot, J)$ monotone
  - $I \subseteq I' \quad \Rightarrow \quad \Phi_P(I, J) \subseteq \Phi_P(I', J)$
- $\Phi_P(I, \cdot)$ antimonotone
  - $J \subseteq J' \quad \Rightarrow \quad \Phi_P(I, J') \subseteq \Phi_P(I, J)$
- $\Phi_P(I, I) = T_P(I)$

Multivalued semantics: 4-val interpretations

Pairs (I, J) of 2-val interpretations

- Atoms in I are known and atoms in J are possible
- Give rise to 4 truth values
  - If $a \in I \cap J$, a is true
  - If $a \notin I \cup J$, a is false
  - If $a \in J \setminus I$, a is unknown
  - If $a \in I \setminus J$, a is overdefined (inconsistent)
- (I, J) consistent if $I \subseteq J$

Alternatively

- Functions val from At to $\{\mathbf{t}, \mathbf{f}, \mathbf{u}, \mathbf{i}\}$
- $I := \{a \mid val(a) = \mathbf{t}$ or $val(a) = \mathbf{i}\}$
- $J := \{a \mid val(a) = \mathbf{t}$ or $val(a) = \mathbf{u}\}$

Multivalued semantics: 4-val interpretations

Pairs (I, J) of 2-val interpretations

- Atoms in I are known and atoms in J are possible
- Give rise to 4 truth values
  - If $a \in I \cap J$, a is true
  - If $a \notin I \cup J$, a is false
  - If $a \in J \setminus I$, a is unknown
  - If $a \in I \setminus J$, a is overdefined (inconsistent)
- (I, J) consistent if $I \subseteq J$

Alternatively

- Functions *val* from *At* to $\{\mathbf{t}, \mathbf{f}, \mathbf{u}, \mathbf{i}\}$
- $I := \{a \mid val(a) = \mathbf{t} \text{ or } val(a) = \mathbf{i}\}$
- $J := \{a \mid val(a) = \mathbf{t} \text{ or } val(a) = \mathbf{u}\}$

Multivalued semantics

4-val one-step provability operator

- $\mathcal{T}_P(I,J) = (\Phi_P(I,J), \Phi_P(J,I))$
- Precision (information) ordering:
  $(I,J) \leq_i (I',J')$ - if $I \subseteq I'$ and $J' \subseteq J$
- $\mathcal{T}_P$ monotone wrt $\leq_i$
- $(I,J) \leq_i (I'J') \quad \Rightarrow \quad \mathcal{T}_P(I,J) \leq_i \mathcal{T}_P(I',J')$
  - We have: $I \subseteq I'$ and $J' \subseteq J$
  - $\Phi_P(I,J) \subseteq \Phi_P(I',J)$ (monotonicity of $\Phi_P(\cdot,J)$)
  - $\Phi_P(I,J') \subseteq \Phi_P(I,J)$ (antimonotonicity of $\Phi_P(I,\cdot)$)

(I,J) consistent $\Rightarrow$ $\mathcal{T}_P(I,J)$ consistent

- Let $I \subseteq J$
- $\Rightarrow \quad \Phi_P(I,J) \subseteq \Phi_P(I,I) \subseteq \Phi_P(J,I)$

Multivalued semantics

4-val one-step provability operator

- $\mathcal{T}_P(I, J) = (\Phi_P(I, J), \Phi_P(J, I))$
- Precision (information) ordering:
  $(I, J) \leq_i (I', J')$ - if $I \subseteq I'$ and $J' \subseteq J$
- $\mathcal{T}_P$ monotone wrt $\leq_i$
- $(I, J) \leq_i (I' J') \quad \Rightarrow \quad \mathcal{T}_P(I, J) \leq_i \mathcal{T}_P(I', J')$
  - We have: $I \subseteq I'$ and $J' \subseteq J$
  - $\Phi_P(I, J) \subseteq \Phi_P(I', J)$ (monotonicity of $\Phi_P(\cdot, J)$)
  - $\Phi_P(I, J') \subseteq \Phi_P(I, J)$ (antimonotonicity of $\Phi_P(I, \cdot)$)

(I, J) consistent $\Rightarrow$ $\mathcal{T}_P(I, J)$ consistent

- Let $I \subseteq J$
- $\Rightarrow \quad \Phi_P(I, J) \subseteq \Phi_P(I, I) \subseteq \Phi_P(J, I)$

4-val supported models

Recall: $\mathcal{T}_P(I,J) = (\Phi_P(I,J), \Phi_P(J,I))$ and $T_P(I) = \Phi_P(I,I)$

- (I,J) is a 4-val supported model of P if $(I,J) = \mathcal{T}_P(I,J)$
- (I,I) is a 4-val supported model iff I is a supported model
  - $(I,I) = \mathcal{T}_P(I,I)$ iff $(I,I) = (\Phi_P(I,I), \Phi_P(I,I)) = (T_P(I), T_P(I))$
- The least 4-val supported model exists!
  - $\mathcal{T}_P$ is monotone and so has the least (wrt $\leq_i$) fixpoint
  - Moreover, it is consistent!
- Kripke-Kleene (Fitting) fixpoint or semantics: $(KK^t(P), KK^p(P))$

- 4-val Gelfond-Lifschitz operator
- $\mathcal{GL}_P(I, J) = (GL_P(J), GL(I))$
- Also monotone wrt $\leq_i$
- (I, J) is a 4-val stable model if $\mathcal{GL}_P(I, J) = (I, J)$
- M is a stable model of P if and only if (M, M) is a 4-val stable model of P
- The least fixpoint of $\mathcal{GL}$ exists!! (by monotonicity)
- And is consistent
  - if $I \subseteq J$ then: $GL_P(J) \subseteq GL(I)$ (antimonotonicity)
- Well-founded fixpoint (semantics): $(WF^t(P), WF^p(P))$
- For every stable model M of P

$$WF^t(P) \subseteq M \subseteq WF^p(P)$$

Logic here-and-there

Logic here-and-there, Heyting 1930

Syntax

- Connectives: $\bot$, $\vee$, $\wedge$, $\rightarrow$
- Formulas - standard extension of atoms by means of connectives
- $\neg\varphi$ - shorthand for $\varphi \rightarrow \bot$
- $\varphi \leftrightarrow \psi$ - shorthand for $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$
- Language $\mathcal{L}_{ht}$

Logic here-and-there

Why important?

- Disjunctive logic programs — special theories in $\mathcal{L}_{ht}$
  - $a_1 | \ldots | a_k \leftarrow b_1, \ldots, b_m, not\ c_1, \ldots not\ c_n$
  - $b_1 \wedge \ldots \wedge b_m \wedge \neg c_1 \wedge \ldots \wedge \neg c_n \rightarrow c_1 \vee \ldots \vee c_n$
- General logic programs (Ferraris, Lifschitz) $=$ theories in $\mathcal{L}_{ht}$
  - answer-set semantics extends to general logic programs
  - equilibrium models in logic *ht*
  - the two coincide!

Entailment in logic here-and-there

Ht-interpretations

- Pairs $\langle H, T \rangle$, where $H \subseteq T$ are sets of atoms
- Kripke interpretations with two worlds "here" and "there"
  - H determines the valuation for "here"
  - T determines the valuation for "there"

Kripke-model satisfiability in the world "here" $\models_{ht}$

- $\langle H, T \rangle \not\models_{ht} \bot$
- $\langle H, T \rangle \models_{ht} p$ if $p \in H$ (for atoms only)
- $\langle H, T \rangle \models_{ht} \varphi \wedge \psi$ and $\langle H, T \rangle \models_{ht} \varphi \vee \psi$ — standard recursion
- $\langle H, T \rangle \models_{ht} \varphi \rightarrow \psi$ if
  - $\langle H, T \rangle \not\models_{ht} \varphi$ or $\langle H, T \rangle \models_{ht} \psi$
  - $T \models \varphi \rightarrow \psi$ (in standard propositional logic).

Entailment in logic here-and-there

Ht-interpretations

- Pairs $\langle H, T \rangle$, where $H \subseteq T$ are sets of atoms
- Kripke interpretations with two worlds "here" and "there"
  - H determines the valuation for "here"
  - T determines the valuation for "there"

Kripke-model satisfiability in the world "here" $\models_{ht}$

- $\langle H, T \rangle \not\models_{ht} \bot$
- $\langle H, T \rangle \models_{ht} p$ if $p \in H$ (for atoms only)
- $\langle H, T \rangle \models_{ht} \varphi \wedge \psi$ and $\langle H, T \rangle \models_{ht} \varphi \vee \psi$ — standard recursion
- $\langle H, T \rangle \models_{ht} \varphi \rightarrow \psi$ if
  - $\langle H, T \rangle \not\models_{ht} \varphi$ or $\langle H, T \rangle \models_{ht} \psi$
  - $T \models \varphi \rightarrow \psi$ (in standard propositional logic).

Entailment in logic here-and-there

ht-model, *ht*-validity, *ht*-equivalence

- If $\langle H, T\rangle \models_{ht} \varphi$ - $\langle H, T\rangle$ is an ***ht-model*** of φ
- φ is ***ht-valid*** if for every *ht*-model $\langle H, T\rangle$, $\langle H, T\rangle \models \varphi$
- φ and ψ are ***ht-equivalent*** if they have the same $ht-$models

- φ and ψ are ht-equivalent iff $\varphi \leftrightarrow \psi$ is *ht*-valid

Proof theory

Natural deduction — sequents and rules

- Sequents $\Gamma \Rightarrow \varphi$ — "φ under the assumptions Γ"
- Introduction rules for $\wedge$, $\vee$, $\rightarrow$

$$\frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \psi}{\Gamma, \Delta \Rightarrow \varphi \wedge \psi}$$

- Elimination rules for $\wedge$, $\vee$, $\rightarrow$

$$\frac{\Gamma \Rightarrow \varphi \quad \Delta \Rightarrow \varphi \rightarrow \psi}{\Gamma, \Delta \Rightarrow \psi}$$

- Contradiction

$$\frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow \varphi}$$

- Weakening

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma' \Rightarrow \varphi} \qquad \text{for all } \Gamma', \Gamma \text{ s.t. } \Gamma' \subseteq \Gamma$$

Proof theory

Axiom schemas

(AS1)	$\varphi \Rightarrow \varphi$	
(AS2)	$\Rightarrow \varphi \vee \neg\varphi$	(Excluded Middle)
(AS2′)	$\Rightarrow \neg\varphi \vee \neg\neg\varphi$	(Weak EM)
(AS2″)	$\Rightarrow \varphi \vee (\varphi \rightarrow \psi) \vee \neg\psi$	(in between (AS2) and (AS2′))

Logics through natural deduction

Propositional logic	(AS1), (AS2)
Intuitionistic logic	(AS1)
Logic here-and-there	(AS1),(AS2″)

Proof theory

Axiom schemas

(AS1)	$\varphi \Rightarrow \varphi$	
(AS2)	$\Rightarrow \varphi \vee \neg\varphi$	(Excluded Middle)
(AS2′)	$\Rightarrow \neg\varphi \vee \neg\neg\varphi$	(Weak EM)
(AS2″)	$\Rightarrow \varphi \vee (\varphi \rightarrow \psi) \vee \neg\psi$	(in between (AS2) and (AS2′))

Logics through natural deduction

Propositional logic	(AS1), (AS2)
Intuitionistic logic	(AS1)
Logic here-and-there	(AS1),(AS2″)

Bringing the two together

Soundness and completeness

- A formula is a theorem of *ht* if and only if it is *ht*-valid

In particular

- φ and ψ are *ht*-equivalent iff $\Rightarrow \varphi \leftrightarrow \psi$ is a theorem of *ht*

Bringing the two together

Soundness and completeness

- A formula is a theorem of *ht* if and only if it is *ht*-valid

In particular

- φ and ψ are *ht*-equivalent iff $\Rightarrow \varphi \leftrightarrow \psi$ is a theorem of *ht*

Logic here-and-there and ASP

Equilibrium models, Pearce 1997

- $\langle T, T \rangle$ is an *equilibrium model* of a set A of formulas if
  - $\langle T, T \rangle \models_{ht} A$, and
  - for every $H \subseteq T$ such that $\langle H, T \rangle \models_{ht} A$, $H = T$

Key connection

- A set M of atoms is an answer set of a disjunctive logic program P (general logic program P) if and only if $\langle M, M \rangle$ is an equlibrium model for P

Logic here-and-there and ASP

Equilibrium models, Pearce 1997

- $\langle T, T\rangle$ is an *equilibrium model* of a set A of formulas if
  - $\langle T, T\rangle \models_{ht} A$, and
  - for every $H \subseteq T$ such that $\langle H, T\rangle \models_{ht} A$, $H = T$

Key connection

- A set M of atoms is an answer set of a disjunctive logic program P (general logic program P) if and only if $\langle M, M\rangle$ is an equlibrium model for P

Key application

Strong equivalence

- Let P and Q be two (general) programs. The following conditions are equivalent:
  - P and Q are strongly equivalent
  - P and Q are *ht*-equivalent
  - P and Q have the same *ht*-models
  - $P \leftrightarrow Q$ is *ht*-valid
  - $\Rightarrow P \leftrightarrow Q$ is a theorem of *ht*

Algebraic approach

The problem

Complex landscape of nonmonotonicity

- Multitude of formalisms
- Different intuitions
- Different languages
- Different semantics
- Complexity

Needed!

- Unifying abstract foundation

The problem

Complex landscape of nonmonotonicity

- Multitude of formalisms
- Different intuitions
- Different languages
- Different semantics
- Complexity

Needed!

- Unifying abstract foundation

A triumph of universal algebra

Basic lesson for this segment

- Major nonmonotonic systems
  - logic programming
  - default logic
  - autoepistemic logics

  can be given a unified algebraic treatment
- Each system can be assigned the same family of semantics
- Key concepts: lattices and bilattices, operators and fixpoints
- Key ideas: approximating operators and stable operators
- Key tool: Knaster-Tarski Theorem

Overview of approach

Generalize Fitting's work on logic programming

- Central role of 4-valued van Emden-Kowalski operator $\mathcal{T}_P$
- Derived stable operator, Ψ'_P
- 2-valued and 3-valued supported models and Kripke-Kleene semantics described by fixpoints of $\mathcal{T}_P$
- 2-valued and 3-valued stable models and well-founded semantics described by fixpoints of Ψ'_P

Lattices

Key definitions, some notation

- $\langle L, \leq \rangle$
  - L is a nonempty set
  - $\leq$ is a partial order such that every two lattice elements have *lub* (join) and *glb* (meet)
- Elements of L express
  - degree of truth
  - measure of knowledge
- $\leq$ - order of increased truth or knowledge
- Complete lattices (both bounds defined for all sets)
- $\perp$, $\top$

Lattices - examples

Lattice $\mathcal{TWO}$

- $\{\mathbf{f}, \mathbf{t}\}$
- $\mathbf{f} \leq \mathbf{t}$

Lattice $\mathcal{A}_2$

- set of all 2-valued interpretations
- componentwise extension of the ordering from $\mathcal{TWO}$

Lattice $\mathcal{W}$

- family of sets of 2-valued interpretations
- $W_1 \sqsubseteq W_2$ if $W_2 \subseteq W_1$

Lattices - examples

Lattice $\mathcal{TWO}$

- $\{\mathbf{f}, \mathbf{t}\}$
- $\mathbf{f} \leq \mathbf{t}$

Lattice $\mathcal{A}_2$

- set of all 2-valued interpretations
- componentwise extension of the ordering from $\mathcal{TWO}$

Lattice $\mathcal{W}$

- family of sets of 2-valued interpretations
- $W_1 \sqsubseteq W_2$ if $W_2 \subseteq W_1$

Lattices - examples

Lattice $\mathcal{TWO}$

- $\{\mathbf{f}, \mathbf{t}\}$
- $\mathbf{f} \leq \mathbf{t}$

Lattice $\mathcal{A}_2$

- set of all 2-valued interpretations
- componentwise extension of the ordering from $\mathcal{TWO}$

Lattice $\mathcal{W}$

- family of sets of 2-valued interpretations
- $W_1 \sqsubseteq W_2$ if $W_2 \subseteq W_1$

Operators

That's what it's all about!

- Truth or knowledge can be revised
- Revisions are described by operators on lattices
- Fixpoints — states of truth or knowledge that cannot be revised

Operators

Monotone operators

- An operator O is monotone if $x \leq y$ implies $O(x) \leq O(y)$
- Knaster-Tarski Theorem: a monotone operator on a complete lattice has a least fixpoint

Operators, cont'd

Antimonotone operators

- An operator O is antimonotone if $x \leq y$ implies $O(y) \leq O(x)$
- If O is antimonotone then O^2 is monotone:

$$x \leq y \quad \Rightarrow \quad O(y) \leq O(x) \quad \Rightarrow \quad O^2(x) \leq O^2(y)$$

- Oscillating pair: (x, y) is an oscillating pair for an operator O if $O(x) = y$ and $O^2(x) = x$
- Antimonotone operator O has an *extreme* oscillating pair

$$(lfp(O^2), gfp(O^2))$$

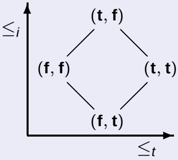
Approximations and bilattices

Key definitions, some notation

- A pair (x, y) approximates an element z if $x \leq z \leq y$
- Orderings of approximations:
  - information (or precision) ordering: $(x_1, y_1) \leq_i (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_2 \leq y_1$
  - truth ordering: $(x_1, y_1) \leq_t (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$
- Bilattice $\langle L^2, \leq_i, \leq_t \rangle$
- A pair (x, y) is consistent if $x \leq y$, and inconsistent, otherwise
- An element (x, y) is complete if $x = y$

Bilattices - examples

Bilattice $\mathcal{FOUR}$

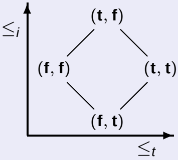


Bilattice $\mathcal{A}_4$

- set of all pairs of 2-valued interpretations (identified with 4-valued interpretations)
- componentwise extension of the orderings from $\mathcal{FOUR}$

Bilattices - examples

Bilattice $\mathcal{FOUR}$



Bilattice $\mathcal{A}_4$

- set of all pairs of 2-valued interpretations (identified with 4-valued interpretations)
- componentwise extension of the orderings from $\mathcal{FOUR}$

Bilattices - examples, cont'd

Bilattice $\mathcal{B}$

- Family of pairs of sets of 2-valued interpretations
- *Belief pairs*
- $(P_1, S_1) \sqsubseteq_i (P_2, S_2)$ if $P_2 \subseteq P_1$ and $S_1 \subseteq S_2$
- $(P_1, S_1) \sqsubseteq_t (P_2, S_2)$ if $P_2 \subseteq P_1$ and $S_2 \subseteq S_1$

Approximating operators

Key definitions, some notation

- $A : L^2 \rightarrow L^2$ approximates $O : L \rightarrow L$ if
  - $A(x,x) = (O(x), O(x))$
  - A is $\leq_i$-monotone
  - A is symmetric: $A^1(x,y) = A^2(y,x)$, where $A(x,y) = (A^1(x,y), A^2(x,y))$

Properties

- Approximating operators are consistent
- Complete fixpoints of A correspond to fixpoints of O
- Every fixpoint of A is approximated by the least fixpoint of A: Kripke-Kleene fixpoint of A
- Kripke-Kleene fixpoint of an approximating operator is consistent

Approximating operators

Key definitions, some notation

- $A : L^2 \to L^2$ approximates $O : L \to L$ if
  - $A(x,x) = (O(x), O(x))$
  - A is $\leq_i$-monotone
  - A is symmetric: $A^1(x,y) = A^2(y,x)$, where $A(x,y) = (A^1(x,y), A^2(x,y))$

Properties

- Approximating operators are consistent
- Complete fixpoints of A correspond to fixpoints of O
- Every fixpoint of A is approximated by the least fixpoint of A: Kripke-Kleene fixpoint of A
- Kripke-Kleene fixpoint of an approximating operator is consistent

Getting down to business!

Stable operators

- If $A : L^2 \to L^2$ is $\leq_i$-monotone then $A^1(\cdot, y)$ and $A^2(x, \cdot)$ are monotone
- For $\leq_i$-monotone operator $A : L^2 \to L^2$ define:

$$C_A^l(y) = \mathit{lfp}(A^1(\cdot, y)) \quad \text{and} \quad C_A^u(x) = \mathit{lfp}(A^2(x, \cdot))$$

- Since A is symmetric, $C_A^l = C_A^u = C_A$
- Stable operator for A:

$$\mathcal{C}_A(x, y) = (C_A(y), C_A(x))$$

- Stable fixpoints (relative to $\mathcal{C}_A$)
- $\leq_i$-least fixpoint of $\mathcal{C}_A$ — well-founded (WF) fixpoint of A

Properties of stable operators

All quite easy to prove, in fact

- C_A is antimonotone
- $\mathcal{C}_A$ is $\leq_i$-monotone and $\leq_t$-antimonotone
- Fixpoints of $\mathcal{C}_A$ are $\leq_t$-minimal fixpoints of A
- Complete fixpoints of $\mathcal{C}_A$ correspond to fixpoints of C_A
- Complete fixpoints of $\mathcal{C}_A$ are fixpoints of O
- K-K fixpoint of $A \leq_i$ WF fixpoint of A

Logic programming — case study 1

Fitting

- Lattice $\mathcal{A}_2$, bilattice $\mathcal{A}_4$
- Operators associated with program P
  - 2-valued van Emden-Kowalski operator T_P
  - Its approximation: 4-valued van Emden-Kowalski operator $\mathcal{T}_P$
  - 2-valued stable operator (Gelfond-Lifschitz operator GL_P)
  - Stable operator $\mathcal{C}_P$ of $\mathcal{T}_P$ (operator Ψ'_P of Przymusinski)
- Semantics
  - Supported models: fixpoints of the operator $\mathcal{T}_P$ (T_P)
  - Kripke-Kleene semantics: least fixpoint of $\mathcal{T}_P$
  - Stable models: fixpoints of the operator $\mathcal{C}_P$ (C_P)
  - Well-founded semantics: least fixpoint of $\mathcal{C}_P$

Logic programming explained

Central role of $\mathcal{T}_P$

$$
\begin{tikzcd}
 & \mathcal{T}_P \arrow[dl] \arrow[dr] & \\
T_P & & \mathcal{C}_P \arrow[dl, leftrightarrow] \\
 & C_P &
\end{tikzcd}
$$

Autoepistemic Logic — case study 2

Truth assignment function $\mathcal{H}_{V,I}$

- For atom p: $\mathcal{H}_{V,I}(p) = I(p)$
- The boolean connectives — standard way
- $\mathcal{H}_{V,I}(KF) = \mathbf{t}$, if for every $J \in V$, $\mathcal{H}_{V,J}(F) = \mathbf{t}$
- $\mathcal{H}_{V,I}(KF) = \mathbf{f}$, otherwise

AE models, expansions

- Moore's operator $D_T : \mathcal{W} \to \mathcal{W}$

$$D_T(V) = \{I : \mathcal{H}_{V,I}(T) = \mathbf{t}\}$$

- Fixpoints of D_T — autoepistemic models of T
- Autoepistemic models generate *expansions*

Autoepistemic Logic — case study 2

Truth assignment function $\mathcal{H}_{V,I}$

- For atom p: $\mathcal{H}_{V,I}(p) = I(p)$
- The boolean connectives — standard way
- $\mathcal{H}_{V,I}(KF) = \mathbf{t}$, if for every $J \in V$, $\mathcal{H}_{V,J}(F) = \mathbf{t}$
- $\mathcal{H}_{V,I}(KF) = \mathbf{f}$, otherwise

AE models, expansions

- Moore's operator $D_T : \mathcal{W} \to \mathcal{W}$

$$D_T(V) = \{I : \mathcal{H}_{V,I}(T) = \mathbf{t}\}$$

- Fixpoints of D_T — autoepistemic models of T
- Autoepistemic models generate *expansions*

AEL — approximating operators

The setting

- Lattice $\mathcal{W}$, bilattice $\mathcal{B}$
- $\mathcal{H}^4_{(V,V'),I}$
- Approximating operator for D_T — $\mathcal{D}_T$ (DMT 98)

$$\mathcal{D}_T(V,V') = (\{I : \mathcal{H}^4_{(V,V'),I}(T) \geq_t (\mathbf{f},\mathbf{t})\}, \{I : \mathcal{H}^4_{(V,V'),I}(T) \geq_t (\mathbf{t},\mathbf{f})\})$$

- Complete fixpoints of $\mathcal{D}_T$ — autoepistemic models of T
- The least fixpoint of $\mathcal{D}_T$ — Kripke-Kleene fixpoint
  - approximates all autoepistemic models of T
- The stable operator for $\mathcal{D}_T$: $\mathcal{C}_T(V,V') = (C_T(V'), C_T(V))$
- What are the fixpoints of C_T?

Autoepistemic logic explained

Central role of $\mathcal{D}_T$

$$
\begin{tikzcd}
 & \mathcal{D}_T \arrow[dl] \arrow[dr] & \\
D_T & & \mathcal{C}_T \arrow[dl, leftrightarrow] \\
 & C_T &
\end{tikzcd}
$$

Default Logic — case study 3

Same setting as for AEL

- Lattice $\mathcal{W}$, bilattice $\mathcal{B}$
- $\mathcal{H}_{V,I}(\varphi) = I(\varphi)$, for every formula φ
- $d = \frac{\alpha :\, \beta_1, \ldots, \beta_k}{\gamma}$
- $\mathcal{H}_{V,I}(d) = \mathbf{t}$ iff
  - there is $J \in V$ such that $J(\alpha) = \mathbf{f}$, or
  - there is i, $1 \leq i \leq k$ such that for every $J \in V$, $J(\beta_i) = \mathbf{f}$, or
  - $I(\gamma) = \mathbf{t}$
- Weak-extension operator E_Δ (Δ — default theory):

$$E_\Delta(V) = \{I \in \mathcal{A}_2 \colon \mathcal{H}_{V,I}(\Delta) = \mathbf{t}\}$$

- Fixpoints of $E_\Delta(V)$ — default models of weak extensions of Δ

4-valued truth assignment, approximating operator

- $\mathcal{H}^4_{(V,V'),I}$
- Approximating operator for E_Δ — $\mathcal{E}_\Delta$

$$\mathcal{E}_\Delta(V,V') = (\{I : \mathcal{H}^4_{(V,V'),I}(\Delta) \geq_t (\mathbf{f},\mathbf{t})\}, \{I : \mathcal{H}^4_{(V,V'),I}(\Delta) \geq_t (\mathbf{t},\mathbf{f})\})$$

- Complete fixpoints of $\mathcal{E}_\Delta$ — models of weak extensions of Δ
- The least fixpoint of $\mathcal{E}_\Delta$ — Kripke-Kleene fixpoint
  - approximates all default models of weak extensions of Δ

Stable operator

- The stable operator for $\mathcal{E}_\Delta$:

$$\mathcal{C}_\Delta(V, V') = (C_\Delta(V'), C_\Delta(V))$$

- C_Δ — Guerreiro-Casanova operator Σ_Δ
- Fixpoints of C_Δ — default models of Reiter's extensions
- Consistent fixpoints of $\mathcal{C}_\Delta$ — stationary extensions by Przymusinski
- Well-founded fixpoint of $\mathcal{E}_\Delta$ (least fixpoint of $\mathcal{C}_\Delta$ — well-founded semantics of default logic by Baral and Subrahmanian)

DL explained

Central role of $\mathcal{E}_\Delta$

$$
\begin{tikzcd}
& \mathcal{E}_\Delta \arrow[dl] \arrow[dr] & \\
E_\Delta & & \mathcal{C}_\Delta \\
& C_\Delta \arrow[ur, leftrightarrow] &
\end{tikzcd}
$$

Connections

Strong parallels!

$$
\begin{tikzcd}
& \mathcal{T}_P \arrow[dl] \arrow[dr] & & & & \mathcal{E}_\Delta \arrow[dl] \arrow[dr] & \\
T_P & & \mathcal{C}_P \arrow[dl, leftrightarrow] & & E_\Delta & & \mathcal{C}_\Delta \arrow[dl, leftrightarrow] \\
& C_P & & & & C_\Delta &
\end{tikzcd}
$$

$$\longrightarrow$$

$$c \leftarrow a, not\ b \quad \Rightarrow \quad \frac{a : \neg b}{c}$$

Connections

Strong parallels!

$$\begin{tikzcd}
 & \mathcal{T}_P \arrow[dl] \arrow[dr] & \\
T_P & & \mathcal{C}_P \arrow[dl, leftrightarrow] \\
 & C_P &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{E}_\Delta \arrow[dl] \arrow[dr] & \\
E_\Delta & & \mathcal{C}_\Delta \arrow[dl, leftrightarrow] \\
 & C_\Delta &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{D}_T \arrow[dl] \arrow[dr] & \\
D_T & & \mathcal{C}_T \arrow[dl, leftrightarrow] \\
 & C_T &
\end{tikzcd}$$

$$\longrightarrow \qquad\qquad \longrightarrow$$

$$c \leftarrow a, not\ b \quad \Rightarrow \quad \frac{a : \neg b}{c} \qquad\qquad \frac{\alpha : \beta}{\gamma} \quad \Rightarrow \quad K\alpha \wedge \neg K \neg \beta \supset \gamma$$

Thank you!