May 2022 Discrete Mathematics Qualifying Exam

Closed book, closed notes, no calculators.

1) Induction. (22 pts) Prove using mathematical induction that

$$\sum_{i=2}^{n}(4i-3) = (2n+1)(n-1) \text{ for all integers } n \geq 2.$$

Note: You have to use mathematical induction; if you prove the statement using any other method, it will not count.

2) Logic.

(a) (10 pts) Let A, B, C be the following statements:

A: Today is Tuesday

B: Mary is at work

C: Today is a holiday

Translate the following English statements into propositional logic:

i. Mary is at work only if today is Tuesday and not a holiday.

ii. Mary not being at work is a necessary condition for today being a holiday.

(b) (9 pts) Determine whether the following logical expressions are logically equivalent or not: $(p \rightarrow q) \wedge (r \rightarrow q)$ and $(p \wedge r) \rightarrow q$. Justify your answer.

3) Sets. Recall that $X \in Y$ means that X is an element of set Y (X can be an object or a set). $X \subseteq Y$ means that set X is a subset of set Y. $X \subset Y$ means that set X is a proper subset of set Y. $\varnothing$ denotes an empty set (a set with no elements). Which of the following statements are true for all sets A, B, and C?

(a) (3 pts) $\varnothing \in A$.

(b) (3 pts) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.

(c) (3 pts) If $A \neq B$ and $B \neq C$, then $A \neq C$.

(d) (3 pts) If $A \in B$ and B is not a subset of C, then $A \notin C$.

(e) (3 pts) $(A - B) \cap (B - A) = \varnothing$.

(f) (3 pts) $A \times B = B \times A$.

(g) (3 pts) $\varnothing \times A = \varnothing$.

4) Functions.

(a) Let S={2,3,5,7} and T={4,6,8}.

i. (6 points) Give an example of a function from S to T that is onto.

ii. (6 points) How many different functions from S to T are there?

iii. (6 points) How many 1-to-1 functions from S to T are there?

(b) (6 points) Modular arithmetic. Give an integer value for the following expression. You should not need a calculator.

$$(431^{12} + 19) \bmod 10$$

5) Proofs. (14 pts) Give a proof by contradiction of the following statement:

Among any group of 25 people, at least three of the chosen people have the same birth month.