Qual Exam (Spring 2015) Automata
Answer all questions. Closed book exam.

Question 1 (10% + 25%)

Let $L^{\frac{1}{2}} = \{w \mid ww \in L\}$.

(a) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Using the notations $Q, \Sigma, \delta, q_0, F$, give the design of an NFA M' such that $L(M') = L(M)^{\frac{1}{2}}$.

Answer (Sketch): For $q \in Q$, we define a DFA $M_q = (Q \times Q, \Sigma, \delta', (q_0, q), \{q\} \times F)$ where $\delta'((q_1, q_2), a) = (\delta(q_1, a), \delta(q_2, a))$. Then M' is the nondeterministic union of DFAs $\{M_q \mid q \in Q\}$.

(b) Let $L = \{a^n b a^n b \mid n \geq 0\}$. Is $(L^3)^{\frac{1}{2}}$ regular? context-free? Justify your answer. Note that $L^3 = \{xyz \mid x, y, z \in L\}$.

Answer (Sketch): $(L^3)^{\frac{1}{2}} = \{a^n b a^n b a^n b \mid n \geq 0\}$ is not context-free, which proof using pumping lemma is similar to that for $\{a^n b^n c^n \mid n \geq 0\}$.

Question 2 (25%)

Let $\Sigma = \{a, b\}$. Given a string $w = a_1 a_2 \ldots a_n$ where $a_1, a_2, \ldots, a_n \in \Sigma$, we say that $a_{i_1} a_{i_2} \ldots a_{i_k}$ is a *sample* of w if $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $0 \leq k \leq n$. Given a language L, we define $\texttt{sample}(L) = \{x \mid x$ is a sample of $w \in L\}$. Show that regular languages are closed under `sample` by giving an algorithm that takes any regular expression r and returns a regular expression r' such that $L(r') = \texttt{sample}(L(r))$.

Answer: We assume that $\Sigma = \{a, b\}$.

```
sample( r ) {
   case r of
      ∅:        return ∅
      ϵ:        return ϵ
      a:        return ϵ ∪ a
      b:        return ϵ ∪ b
      r1 ∪ r2:  return sample(r1) ∪ sample(r2)
      r1 · r2:  return sample(r1) · sample(r2)
      r*        return (sample(r))*
}
```

Question 3 (15%)

Given a string $w = a_1a_2\ldots a_n$ where $a_1, a_2, \ldots, a_n \in \Sigma$, we say that $a_{i_1}a_{i_2}\ldots a_{i_n}$ is a *scramble* of w if $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$. Given a language L, we define $\texttt{scramble}(L) = \{x \mid x \text{ is a scramble of } w \in L\}$. Is the class of regular languages closed under `scramble`? Justify your answer.

Answer (Sketch): No. $\texttt{scramble}((01)^*) = \{w \in \{0,1\}^* \mid w \text{ has the same number of 0's and 1's }\}$, which can be shown to be not regular.

Question 4 (25%)

Let $P = \{<M> \mid M \text{ is a TM that accepts } w^R \text{ whenever it accepts } w\}$. Show that P is undecidable.

Answer (Sketch): (*This question is taken from Problem 5.9 of the textbook.*) We can modify the proof to Theorem 5.3 ($\texttt{REGULAR}_{\mathrm{TM}}$ is undecidable). Instead of testing if x has the form 0^n1^n, we test if $x = 01$. Then $L(<M_2>)$ either is $\{0,1\}^*$ (that is, $M_2 \in P$) or $\{01\}$ (that is, $M_2 \notin P$).