Ph.D. Qualifying Exam: Analysis of Algorithms

This is a closed book exam. The total score is 100 points. Please answer all questions.

1. We write $LCS(\alpha, \beta)$ to denote the longest common subsequence of α and β, and $lcs(\alpha, \beta)$ to denote the length of $LCS(\alpha, \beta)$.

(20 points) (a) Suppose there is a function $\textsc{Mid}(x, y)$ that takes two sequences x and y of lengths m and $n > 1$ respectively, and in time $\Theta(mn)$ and space $\Theta(m + n)$ returns an integer i such that

$$lcs(x[1..m], y[1..n]) = lcs(x[1..i], y[1..\lfloor n/2 \rfloor]) + lcs(x[i+1..m], y[\lfloor n/2 \rfloor + 1..n]).$$

Give an efficient algorithm for computing $LCS(x, y)$ in space $o(mn)$ and time $O(mn)$, using function $\textsc{Mid}()$. Analyze the space usage and running time of your algorithm.

(10 points) (b) The function $\textsc{Mid}(x, y)$ can be implemented as

$$\operatorname*{argmax}_{1 \le i \le m} \quad lcs(x[1..i], y[1..\lfloor n/2 \rfloor]) \ + \ lcs(x[i+1..m], y[\lfloor n/2 \rfloor + 1..n])$$

and computed in time and space $O(m)$ provided that

$$lcs(x[1..i], y[1..\lfloor n/2 \rfloor])$$

and

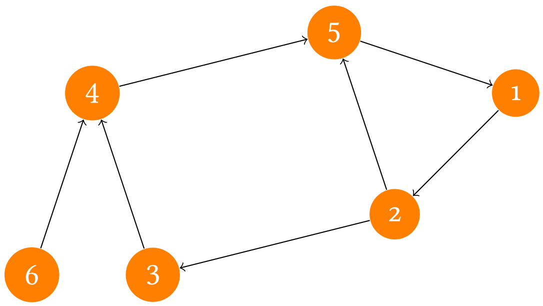
$$lcs(x[i..m], y[\lfloor n/2 \rfloor + 1..n])$$

have been computed for all $i \in [1, m]$.

Show how to compute $lcs(x[1..i], y[1..\lfloor n/2 \rfloor])$ for all $i \in [1, m]$ in time $O(mn)$ and space $O(m + n)$.

(10 points) (c) Show how to compute $lcs(x[i..m], y[\lfloor n/2 \rfloor + 1..n])$ for all $i \in [1, m]$ in time $O(mn)$ and space $O(m + n)$. Hint: $lcs(x, y) = lcs(x^R, y^R)$ where x^R and y^R are the reversals of x and y respectively.

2. In a directed graph, a path is *Eulerian* if the path traverses each edge of the graph exactly once. In this question, we call an Eulerian path an *Euler path* if the path begins and ends with different vertices. An Euler path may visit a vertex multiple times and not be simple. An Euler path may not exist in a graph. Figure 1 shows two directed graphs with and without Euler paths.



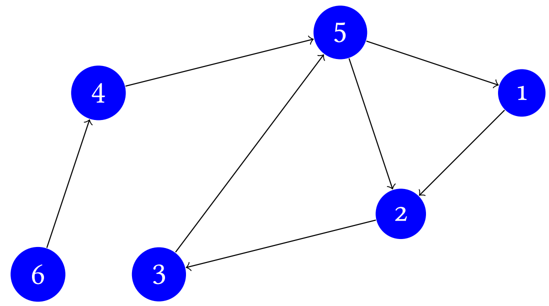


Figure 1: Examples. On the left graph, no Euler path exists; on the right graph, there are two Euler paths: $6 \to 4 \to 5 \to 2 \to 3 \to 5 \to 1 \to 2$ and $6 \to 4 \to 5 \to 1 \to 2 \to 3 \to 5 \to 2$.

(30 points) (a) State and justify (carefully and mathematically) the necessary and sufficient conditions for the existence of an Euler path in a directed graph based on in- and out-degrees of each node. We assume that the graph has an underlying connected undirected graph.

(30 points) (b) Develop an efficient algorithm to compute an Euler path in a directed graph where such a path exists. Give the asymptotic time complexity of your algorithm.