**Automata Qual Exam (Spring 2012)**
Answer ALL questions (Closed Book Exam)

Question 1 (15 points)

For each of the following, either show that it is always true or exhibit a counterexample.
(a) If $L_1 \bigcup L_2$ is regular, then $L_1$ is regular.
(b) If $L_1 \cdot L_2$ is regular, then $L_1$ is regular.
(c) If $L^*$ is regular, then $L$ is regular.

Question 2

Consider the following context-free grammar $G$:

$$S \longrightarrow aaSb \mid aSbb \mid \epsilon$$

Note: $L(G) \subseteq a^*b^*$. Below are the possible $i$ and $j$ such that $a^ib^j \in L(G)$:

| $i$ | $j$ |
|---|---|
| 0 | 0 |
| 1 | 2 |
| 2 | 1, 4 |
| 3 | 3, 6 |
| 4 | 2, 5, 8 |
| 5 | 4, 7, 10 |
| 6 | 3, 6, 9, 12 |
| 7 | 5, 8, 11, 14 |
| 8 | 4, 7, 10, 13, 16 |
| 9 | 6, 9, 12, 15, 18 |
| 10 | 5, 8, 11, 14, 17, 20 |
| 11 | 7, 10, 13, 16, 19, 22 |
| 12 | ... |
| 13 | ... |
| ... | ... |

(a) (15 points)

It is given that $L(G) = \{a^{2n}b^{f(n,k)} \mid 0 \leq k \leq n\} \cup \{a^{2n+1}b^{g(n,k)} \mid 0 \leq k \leq n\}$. What are $f(n,k)$ and $g(n,k)$?

(b) (15 points) Prove that the characterization for $L(G)$ given in part (a) is correct using mathematical induction. Note: you can assume *without proof* that $L(G) \subseteq a^*b^*$.

(c) (10 points) Give a context-free grammar $G'$ such that $L(G') = \{w \mid w \in L(G),\ |w| \text{ is even } \}$.

(d) (10 points) Give a context-free grammar $G''$ such that $L(G'') = \{w \mid w \in L(G),\ |w| \text{ is odd } \}$.

Question 3

(a) (20 points) Explain how a deterministic Turing machine can simulate a nondeterministic Turing machine for recognizing the same language.

(b) (15 points) Suppose we modify the definition of nondeterministic Turing machine so that a string is accepted if the string is accepted by every possible computation path. (In contrast, a normal nondeterministic Turing machine accepts a string $w$ if there exists one accepting path that accepts $w$.) Explain how a deterministic Turing machine can simulate a nondeterministic Turing machine according to the modified definition.