December 2021 Discrete Mathematics Qualifying Exam

Closed book, closed notes, no calculators.

1) (22 points) Prove using mathematical induction that

$$\sum_{i=1}^{n}(3i+8) = (3n+19)n/2 \text{ for all integers } n \geq 1.$$

Note: You have to use mathematical induction; if you prove the statement using any other method, it will not count.

Solution:

Proof (by mathematical induction):

Base case:

The statement is true for n=1 because LHS = $3+8=11$ and RHS = $\frac{(3+19)}{2} = 11$.

Inductive step:

Suppose that for some integer $k \geq 1, \sum_{i=1}^{k}(3i+8) = \frac{(3k+19)k}{2}$. This is the inductive hypothesis.

We must show that $\sum_{i=1}^{k+1}(3i+8) = \frac{(3(k+1)+19)(k+1)}{2}$.

LHS = $\sum_{i=1}^{k+1}(3i+8)$

= $\left(\sum_{i=1}^{k}(3i+8)\right) + (3(k+1)+8)$ by separating the last term

= $\frac{(3k+19)k}{2} + (3(k+1)+8)$ by inductive hypothesis

= $\frac{3k^2+19k}{2} + (3k+11) = \frac{3k^2+19k+6k+22}{2} = \frac{3k^2+25k+22}{2}$ by algebra

RHS = $\frac{(3(k+1)+19)(k+1)}{2} = \frac{(3k+22)(k+1)}{2} = \frac{3k^2+22k+3k+22}{2} = \frac{3k^2+25k+22}{2}$.

LHS = RHS. Therefore, the statement is true for all integers $n \geq 1$.

2) (10 points) Use a truth table to show whether the following expression is a tautology, contradiction or neither:

$$(p \vee q) \vee (q \rightarrow p).$$

Solution:

p	q	p ∨ q	q → p	(p ∨ q) ∨ (q → p)
T	T	T	T	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

The statement is a tautology.

3) (10 points) Imagine that num_orders and num_instock are particular values, such as might occur during execution of a computer program. Write negation for the following statement:

(num_orders > 100 and num_instock ≤ 500) or num_instock < 200

Solution:

Negation: (num_orders ≤ 100 or num_instock > 500) and num_instock ≥ 200.

4)

(a) (6 points) Let $A = \{w, x, y, z\}$ and $B = \{a, b\}$. List all the elements of the Cartesian product $A \times B$.

(b) (5 points) Is {{5, 4}, {7, 2}, {1, 3, 4}, {6, 8}} a partition of {1, 2, 3, 4, 5, 6, 7, 8}? Explain your answer.

(a) (5 points) Is {{w, x, v}, {u, y, q}, {p, z}} a partition of {p, q, u, v, w, x, y, z}? Explain your answer.

Solution:

(a) $A \times B = \{ (w, a), (w, b), (x, a), (x, b), (y, a), (y, b), (z, a), (z, b) \}$
(b) No. 4 appears twice.
(c) Yes. Sets {w, x, v}, {u, y, q}, {p, z} are disjoint and {w, x, v}∪{u, y, q}∪{p, z} = {p, q, u, v, w, x, y, z}. Each element from {p, q, u, v, w, x, y, z} appears in exactly one of the sets {w, x, v}, {u, y, q}, {p, z}.

5) For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of the inverse. In (a) and (b) below, **R** is a set of real numbers.

(a) (6 points) Function h: **R**→ **R** where $h(x) = x^2 + 4$.

(b) (6 points) Function g: **R**→ **R** where $g(x) = 2x + 3$

(c) (6 points) Let A be the set {1, 2, 3, 4, 5, 6, 7, 8}. Recall that P(A) denotes the power set of A which is the set of all subsets of A. Function f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8} is defined as follows. For $X \subseteq A$, $f(X) = |X|$, where $|X|$ is the cardinality of X (the number of elements in X).

Solution:

(a) The inverse is not well-defined because the function is not onto as $x^2 + 4$ can not be negative (there are no elements that map to negative numbers).
(b) The inverse is well-defined. If $g(x) = y$, then $y = 2x + 3$, $y - 3 = 2x$, $x = (y-3)/2$. It can be also written as $g^{-1}(x) = (x-3)/2$.
(c) The inverse is not well-defined because f is not one-to-one: e.g., f({1,2}) = 2 and f({2,3})= 2.

6) How many 8-bit strings (strings of 0s and 1s of length 8) are there subject to each of the following restrictions?

(a) (4 points) No restrictions. (That is, how many different bit strings of length 8 are there?)
(b) (4 points) The string starts with 01.
(c) (4 points) The string starts with 001 or 10.
(d) (4 points) The first two bits are the same as the last two bits.

(e) (4 points) The string has exactly three 0's.

(f) (4 points) There is exactly one 1 in the first half and exactly three 1's in the second half.

Solution:

(a) No restrictions. 2^8

(b) The string starts with 01. 2^6

(c) The string starts with 01 or 101. $2^6 + 2^5$

(d) The first two bits are the same as the last two bits. 2^6

(e) The string has exactly three 0's. $\binom{8}{3} = 56$

(f) There is exactly one 1 in the first half and exactly two 1's in the second half. $\binom{4}{1}\binom{4}{2} = 4\binom{4}{2} = 4 \cdot 6 = 24$