December 2019 Discrete Mathematics Qualifying Exam

Closed book, closed notes, no calculators.

1) (15 points) Prove using mathematical induction that

$$\sum_{i=2}^{n}(2i-1) = n^2 - 1 \text{ for all integers } n \geq 2.$$

Note: You have to use mathematical induction; if you prove the statement using any other method, it will not count.

Solution:

Proof (by mathematical induction):
Base case:
The statement is true for n=2 because LHS = 2·2−1= 3 and RHS = $2^2 - 1 = 3$.
Inductive step:
Suppose that for some integer $k \geq 2$, $\sum_{i=2}^{k}(2i-1) = k^2 - 1$. This is the inductive hypothesis.
We must show that $\sum_{i=2}^{k+1}(2i-1) = (k+1)^2 - 1$.
LHS = $\sum_{i=2}^{k+1}(2i-1)$
= $\left(\sum_{i=2}^{k}(2i-1)\right) + (2(k+1)-1)$ by separating the last term
= $k^2 - 1 + (2(k+1)-1)$ by inductive hypothesis
= $k^2 + 2k$ by algebra
RHS = $(k+1)^2 - 1 = k^2 + 2k + 1 - 1 = k^2 + 2k$.
LHS = RHS. Therefore, the statement is true for all integers $n \geq 2$.

2) (14 points) Let X={a, b, c, d, e} and Y={0, 1}.
a) Give an example of a function from X to Y.
b) How many different functions are there from X to Y.

Solution:

a) For example, function f defined as follows: f(a) = 1, f(b) = 0, f(c) = 0, f(d) = 1, f(e)=1.
b) There are 2^5 different functions. There are 2 choices (0 or 1) for the value of f(a), 2 choices for the value of f(b), 2 choices for the value of f(c), 2 choices for the value of f(d), and 2 choices for the value of f(e).

3) (14 points) Evaluate the following summation: $\sum_{i=2}^{n+1} 4^i$.

Solution:

$$\sum_{i=2}^{n+1} 4^i = \sum_{j=0}^{n-1} 4^{j+2} = 4^2 \sum_{j=0}^{n-1} 4^j = 4^2 \frac{4^n - 1}{4-1} = \frac{4^{n+2} - 16}{3}$$

4) (14 points) Let A = {n ∈ Z| n = 5r for some integer r} and B = {m ∈ Z|m = 20s for some integer s}.
   a) Is A ⊆ B? Explain.
   b) Is B ⊆ A? Explain.

Solution:

a) No. A is not a subset of B. For example, 5∈A (because 5=5r for r=1) but 5∉B (because 5 is not divisible by 20).
b) Yes. B ⊆ A. Let m∈B. Then, by definition of B, m = 20s for some integer s. We have: m = 20s = 5(4s)=5r, where r=4s. r is an integer because s and 4 are integers and the product of integers is an integer. By definition of A, m∈A.

5) (14 points) Write truth table for the following statement: (q ⟶ p) ∨ ¬r

Solution:

p	q	r	(q ⟶ p) ∨ ¬r
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

6) (14 points) Prove by contradiction that there is no largest integer.

Solution.

Assume the contrary. Assume, there is a largest integer. Let N be the largest integer. Then, N+1 is also an integer and N+1 > N. This contradicts the assumption that N is the largest integer. Therefore, there is no largest integer.

7) (14 points)
   a) For each expression, give an equivalent expression that does not use the log function:
   - $\log_3 1$
   - $2^{\log_2(2n)}$
   - $4^{\log_2 n}$

b) For each expression, give an equivalent expression that is of the form $\log_3(*)$, where * is an expression with numbers and possibly the variable k.

- $\log_3 3k - \log_3 7 + 2\log_3 k$
- $(\log_2 k)/(\log_2 3)$
- $(\log_2(k^2))/(\log_2 9)$
- $\log_9(k^2)$

Solution.

a)

- $\log_3 1 = 0$
- $2^{\log_2(2n)} = 2n$
- $4^{\log_2 n} = n^2$

b)

- $\log_3 3k - \log_3 7 + 2\log_3 k = \log_3 \frac{3k^3}{7}$
- $(\log_2 k)/(\log_2 3) = \log_3 k$
- $(\log_2(k^2))/(\log_2 9) = \log_3 k$
- $\log_9(k^2) = \log_3 k$