December 2018 Discrete Mathematics Qualifying Exam

Closed book closed notes

1. (25 points) Given the following compound proposition (expression in propositional logic), find a logically equivalent expression that uses only ∧ and ¬ and that has no more than one occurrence of p, no more than one occurrence of q, and no more than one occurrence of r):

$$(r \longrightarrow (q \longleftrightarrow r)) \longrightarrow p$$

Solution:
It could be done using truth tables or using logical equivalences. The following solution uses logical equivalences. (Note: you do not have to name the laws you are using as long as they are used correctly)

$(r \longrightarrow (q \longleftrightarrow r)) \longrightarrow p$
(eliminate biconditional using Conditional identities)
$\equiv (r \longrightarrow ((q \longrightarrow r) \wedge (r \longrightarrow q))) \longrightarrow p$
(eliminate implications using Conditional identities)
$\equiv (\neg (\neg r \vee ((\neg q \vee r) \wedge (\neg r \vee q)))) \vee p$
(move negation inward using De Morgan's laws)
$\equiv (\neg\neg r \wedge \neg((\neg q \vee r) \wedge (\neg r \vee q))) \vee p$
(use Double negation law)
$\equiv (r \wedge \neg((\neg q \vee r) \wedge (\neg r \vee q))) \vee p$
(move negation inward using De Morgan's laws)
$\equiv (r \wedge (\neg (\neg q \vee r) \vee \neg(\neg r \vee q))) \vee p$
(move negation inward using De Morgan's laws)
$\equiv (r \wedge ((\neg\neg q \wedge \neg r) \vee (\neg\neg r \wedge \neg q))) \vee p$
(use Double negation law)
$\equiv (r \wedge ((q \wedge \neg r) \vee (r \wedge \neg q))) \vee p$
(use Distributive law)
$\equiv ((r \wedge (q \wedge \neg r)) \vee (r \wedge (r \wedge \neg q))) \vee p$
(use Commutative and Associative laws)
$\equiv (((r \wedge \neg r) \wedge q) \vee ((r \wedge r) \wedge \neg q)) \vee p$
(use Complement and Idempotent laws)
$\equiv ((F \wedge q) \vee (r \wedge \neg q)) \vee p$
(use Commutative and Domination laws)
$\equiv (F \vee (r \wedge \neg q)) \vee p$
(use Commutative and Identity laws)
$\equiv (r \wedge \neg q) \vee p$
(use De Morgan and Double negation laws to remove ∨)
$\equiv \neg(\neg(r \wedge \neg q) \wedge \neg p)$

2. (25 points) Determine whether the following statement is true or false. Justify your answer with a proof or a counterexample, as appropriate.

   For all real numbers x and y, if $-y^3 + x^2y + 2y + 1 \leq x^3 + xy^2 + 2x$, then $y \leq x$.

Solution: The statement is false. Counterexample: Take x=0 and y=2. Then,

LHS = $-y^3 + x^2y + 2y + 1 = -8 + 0 + 4 + 1 = -3.$

RHS = $x^3 + xy^2 + 2x = 0.$

$-3 \leq 0$ However, $y \nleq x$.

3. (25 points) Consider $\{b_n\}$ defined by $b_1 = 1, b_2 = 5$, and $b_n = b_{n-1} + 2b_{n-2}$ for all integers $n \geq 3$. Prove that for all integers $n \geq 1$, $b_n = 2^n + (-1)^n$.

Proof by strong induction.

Base cases.

When $n = 1$, we have $b_1 = 1$ and $2^1 + (-1)^1 = 2 - 1 = 1$.

When $n = 2$, we have $b_2 = 5$ and $2^2 + (-1)^2 = 4 + 1 = 5$.

Inductive step.

Assume that $b_n = 2^n + (-1)^n$ for all integers n from 1 to k for some integer k $\geq$ 2 (strong inductive hypothesis). We need to show that $b_{k+1} = 2^{k+1} + (-1)^{k+1}$.

Since k $\geq$ 2, we have that k+1 $\geq$ 3 and (k+1) −2 $\geq$ 1.

By definition,

$$
\begin{aligned}
b_{k+1} &= b_k + 2b_{k-1} \\
&= 2^k + (-1)^k + 2(2^{k-1} + (-1)^{k-1}) && \text{by strong inductive hypothesis} \\
&= 2^k + (-1)^k + 2^k + 2(-1)^{k-1} \\
&= 2 \cdot 2^k + (-1)(-1)^{k-1} + 2(-1)^{k-1} \\
&= 2^{k+1} + (-1)^{k-1} \\
&= 2^{k+1} + (-1)^{k+1}
\end{aligned}
$$

This is what we needed to show.

3. Consider a set of strings of length n with alphabet {a, b, c, d}. This is a set of strings of length n , where each of the n symbols is one of {a, b, c, d}. n is a positive integer.
   a) (3 points) How many strings of length n with alphabet {a, b, c, d} are there?
   b) (3 points) How many of strings of length n do not contain a?
   c) (3 points) How many of strings of length n contain at least one a?
   d) (3 points) How many of strings of length n do not contain a and do not contain b?
   e) (13 points) Calculate the number of strings of length n with alphabet {a,b,c,d} such that each of a, b, and c occurs at least once.
      For instance, if n=3, then the number of strings of length 3 with alphabet {a,b,c,d} such that each of a, b, and c occurs at least once is 6 (which is the number of permutations of a, b, and c).
      If n=1 or n=2, then the number of such strings is 0.
      You need to derive a general formula for the number of strings of length n with alphabet {a,b,c,d} that contain each of a, b, and c at least once.

Solution:

a) Total number of n-strings of 4 symbols is 4^n.

b) The number of n-strings that do not contain a is 3^n.

c) The number of n-strings that contain at least one a is $4^n - 3^n$.

d) The number of n-strings that do not contain a and do not contain b is 2^n.

e) Let A be a set of n-strings that contain a. Let B be a set of n-strings that contain b. Let C be a set of n-strings that contain c. Then, $\overline{A}$ is a set of n-strings that do not contain a. $\overline{B}$ is a set of n-strings that do not contain b. $\overline{C}$ is a set of n-strings that do not contain c.

The question is asking for $|A \cap B \cap C|$.

$A \cap B \cap C = \overline{\overline{A} \cup \overline{B} \cup \overline{C}}$. Therefore, $|A \cap B \cap C| = 4^n - |\overline{A} \cup \overline{B} \cup \overline{C}|$.

By the inclusion-exclusion principle, $|\overline{A} \cup \overline{B} \cup \overline{C}| = |\overline{A}| + |\overline{B}| + |\overline{C}| - |\overline{A} \cap \overline{B}| - |\overline{A} \cap \overline{C}| - |\overline{B} \cap \overline{C}| + |\overline{A} \cap \overline{B} \cap \overline{C}|$.

$|\overline{A}| = |\overline{B}| = |\overline{C}| = 3^n$.

$|\overline{A} \cap \overline{B}| = |\overline{A} \cap \overline{C}| = |\overline{B} \cap \overline{C}| = 2^n$.

$|\overline{A} \cap \overline{B} \cap \overline{C}| = 1$.

Therefore,

$|\overline{A} \cup \overline{B} \cup \overline{C}| = 3^n + 3^n + 3^n - 2^n - 2^n - 2^n + 1 = 3^{n+1} - 3 \cdot 2^n + 1$.

The number of n-strings with alphabet {a,b,c,d} such that each of a, b, and c occurs at least once is $|A \cap B \cap C| = 4^n - |\overline{A} \cup \overline{B} \cup \overline{C}| = 4^n - 3^{n+1} + 3 \cdot 2^n - 1$.