Qual Exam (Fall 2014) Discrete Mathematics
Answer all questions. Closed book exam.

Question 1

(a) Show that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ by constructing a truth table.

Answer:

p	∧	(q	∨	r)
T	T	T	T	T
T	T	T	T	F
T	T	F	T	T
T	F	F	F	F
F	F	T	T	T
F	F	T	T	F
F	F	F	T	T
F	F	F	F	F

(p	∧	q)	∨	(p	∧	r)
T	T	T	T	T	T	T
T	T	T	T	T	F	F
T	F	F	T	T	T	T
T	F	F	F	T	F	F
F	F	T	F	F	F	T
F	F	T	F	F	F	F
F	F	F	F	F	F	T
F	F	F	F	F	F	F

(b) Let A, B and C be sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. You are asked to make use of the result from (a). Hint: Your proof should begin as $A \cap (B \cup C) = \{x \mid (x \in A) \wedge (x \in (B \cup C))\}$.

Answer: $A \cap (B \cup C) = \{x \mid (x \in A) \wedge (x \in (B \cup C))\} = \{x \mid (x \in A) \wedge ((x \in B) \vee (x \cup C))\} = \{x \mid ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \cup C))\} = \{x \mid (x \in A) \wedge (x \in B)\} \cup \{x \mid (x \in A) \wedge (x \cup C)\} = (A \cap B) \cup (A \cap C)$

(c) Let B and C be sets. Express $|B \cup C|$ in terms of $|B|$, $|C|$ and $|B \cap C|$.

Answer: $|B \cup C| = |B| + |C| - |B \cap C|$.

(d) Let A, B and C be sets. Express $|A \cup B \cup C|$ in terms of $|A|$, $|B|$, $|C|$, $|A \cap B|$, $|A \cap C|$, $|B \cap C|$ and $|A \cap B \cap C|$. You are asked to derive the expression algebraically based on the results from (b) and (c). Drawing a Venn diagram is not sufficient to answer this question. Hint: Rewrite $|A \cup B \cup C|$ first as $|A \cup (B \cup C)|$. You can assume without proof $(A \cap B) \cap (A \cap C) = A \cap B \cap C$.

Answer: $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)| = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| = |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|) = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap B) \cap (A \cap C)| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

(e) Let $A_1, A_2, \ldots A_n$ be sets. Extending the ideas from part (d), we want to give an expression for $|\bigcup_{1 \leq i \leq n} A_i|$. Which of the following expresions is (or, are) correct? If none of them is correct, you are asked to give your own correct expression using similar notations.

(i) $|\bigcup_{1\le i\le n} A_i| = \sum_{S\subseteq\{1,2,\ldots,n\}} |\bigcap_{i\in S} A_i|$

(ii) $|\bigcup_{1\le i\le n} A_i| = \Pi_{S\subseteq\{1,2,\ldots,n\}} |\bigcap_{i\in S} A_i|$

(iii) $|\bigcup_{1\le i\le n} A_i| = \sum_{S\subseteq\{1,2,\ldots,n\}} |\bigcup_{i\in S} A_i|$

(iv) $|\bigcup_{1\le i\le n} A_i| = \Pi_{S\subseteq\{1,2,\ldots,n\}} |\bigcup_{i\in S} A_i|$

(v) $|\bigcup_{1\le i\le n} A_i| = \sum_{S\subseteq\{1,2,\ldots,n\}} (-1)^{|S|} |\bigcap_{i\in S} A_i|$

(vi) $|\bigcup_{1\le i\le n} A_i| = \Pi_{S\subseteq\{1,2,\ldots,n\}} (-1)^{|S|} |\bigcap_{i\in S} A_i|$

(vii) $|\bigcup_{1\le i\le n} A_i| = \sum_{S\subseteq\{1,2,\ldots,n\}} (-1)^{|S|} |\bigcup_{i\in S} A_i|$

(viii) $|\bigcup_{1\le i\le n} A_i| = \Pi_{S\subseteq\{1,2,\ldots,n\}} (-1)^{|S|} |\bigcup_{i\in S} A_i|$

Answer: None is correct. The correct expression is $\sum_{S\subseteq\{1,2,\ldots,n\}} (-1)^{|S|+1} |\bigcap_{i\in S} A_i|$

Question 2

Let A, B and C be sets. We define functions $f : (A \longrightarrow B) \longrightarrow C$ and $g : A \longrightarrow (B \longrightarrow C)$.

(a) How many different possible f's are there? Express your answer in terms of $|A|, |B|$ and $|C|$.

Answers: $|(A \longrightarrow B) \longrightarrow C| = |C|^{\left(|B|^{|A|}\right)}$

(b) How many different possible g's are there? Express your answer in terms of $|A|, |B|$ and $|C|$.

Answers: $|A \longrightarrow (B \longrightarrow C)| = \left(|C|^{|B|}\right)^{|A|}$.

(c) Are the two answers from (a) and (b) the same? Carefully justify your answer.

Answer: No. Let $|A| = 2, |B| = 2, |C| = 3$. $|C|^{\left(|B|^{|A|}\right)} = 2^{(2^3)} = 2^8$ but $\left(|C|^{|B|}\right)^{|A|} = (2^2)^3 = 4^3 = 2^6$.

Question 3

Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent and 7-cent stamps. Show by mathematical induction that $P(n)$ is true for $n \geq 18$. Hint: carefully determine what the base cases are.

Answer:

Base cases: $P(18)$ is true as $18 = 4 + 2 * 7$. $P(19)$ is true as $19 = 3 * 4 + 7$. $P(20)$ is true as $20 = 5 * 4$. $P(21)$ is true as $21 = 3 * 7$.

Induction hypothesis: $P(n)$ are true for $18 \leq n < k$, where $k \geq 22$.

Induction step: Consider $P(k)$. Since $k \geq 22$, we have $k - 4 \geq 18$. By the induction hypothesis, there exists positive integers x and y such that $k-4 = 4x+7y$ which implies that $k = 4(x+1)+7y$. Therefore, $P(k)$ is true.

Question 4

(a) What is the coefficient of x^{10} in $(x + 1)^{15}$? Simplify and give a number for your answer.

Answer: The coefficient is 15 choose 10, which is the same as 15 choose 5. That is, $\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{2 \cdot 3 \cdot 4 \cdot 5} = 7 \cdot 13 \cdot 3 \cdot 11 = 91 \cdot 33 = 3003$.

(b) Suppose Ann and Bob play the following game. Ann flips three coins while Bob flips two. Ann wins if she has more heads showing than Bob. What is the probability that Ann wins? Assume that the coins are fair.

Answer:

Case 1: If Ann gets three heads, she must win. Probability is $(1/2)^3 = 1/8$.

Case 2: If Ann gets two heads and Bod does not get two heads, Ann wins. Probability is $3 * (1/2)^3 * (1 - 1/4) = 9/32$

Case 3: If Ann gets one head and Bob gets none, Ann wins. Probability is $3 * (1/2)^3 * (1/4) = 3/32$

The probability is the sum of the three probabilities, which is $1/8 + 9/32 + 3/32 = 16/32 = 1/2$.

Question 5

Consider a binary relation $R \subseteq A \times A$. With respect to R, we say that a is connected to b if there exists a string $a_1 a_2 \ldots a_n$ over A (that is, $a_1, \ldots, a_n \in A$) such that $a = a_1$, $a_i R a_{i+1}$ for $i = 1, 2, \ldots, n-1$, and $a_n = b$. Note that a and b may not have to be distinct, and the unit-length string a over A shows that a is connected to a.

We recall some notations for strings. Let x and y be strings over A. We write xy to denote the concatenation of x and y. If X and Y are two set of strings, we write XY to denote $\{xy \mid x \in X, y \in Y\}$. We write X^2 to denote XX. For $k > 2$, we write X^k to denote XX^{k-1}. We also write X^1 to denote X, and X^0 to denote $\{\epsilon\}$ where ϵ is the empty string of length 0.

Next, we define the concept of "regular" set of strings over A inductively as follows: (1) The empty set is regular, (2) For $a \in A$, the set $\{a\}$ with a single string of length 1 is regular, (3) If X and Y are regular sets, then $X \cup Y$, the union of X and Y, is regular, (4) If X and Y are regular sets, then XY is regular, (5) If X and Y are regular, then so is $\bigcup_{k \geq 0} X^k$, which we also denote by the notation X^*.

We want to prove the claim that for any $a, b \in A$, the set of strings connecting a to b is regular. We will prove this result by an induction on the size of A.

Base Case: When $|A| = 1$.

(a) What are the possible R relations? For each possible relation of R, prove that the claim holds.

Answer: Let $A = \{a\}$. There are two possible R's. Case 1: aRa. The set of strings from a to a is $\{a^i \mid i \geq 1\} = \{a\}\{a\}^*$, which is regular. Case 2: aRa does not hold. The set of strings from a to a is $\{a\}$, which is regular.

Induction hypothesis: Suppose the claim is true for any A of size n.

Induction step: Consider A where $|A| = n + 1$ and $n \geq 1$.

Case 1: Consider the set of strings connecting from a to a, where $a \in A$.

Case 1a: Suppose aRa does not hold.

For any $b, c \in A$ where $a \neq b$ and $a \neq c$, let $P_{b,c}$ denote the set of strings connecting b to c without a in it.

(b) Argue that $P_{b,c}$, where $a \neq b$ and $a \neq c$, is regular.

Answer: We consider a reduced relation $R' \subseteq (A - \{a\}) \times (A - \{a\})$ such that $xR'y$ iff xRy, where $x, y \in A - \{a\}$. Then $P_{b,c}$ is the set of strings connecting b to c with respect to R'. By the induction hypothesis, $P_{b,c}$ is regular as $|A - \{a\}| = n$.

(c) Show that the set of strings connecting a to a with respect to R is regular. Hint: You may find the following notations useful: $E = \{e \in A \mid aRe\}$ and $F = \{f \in A \mid fRa\}$.

Answer: The set of strings connecting a to a is $\left(\{a\}\left(\bigcup_{e \in E,\, f \in F} P_{ef}\right)\right)^* \{a\}$, which is regular. In case $E = F = \emptyset$, the set of strings is still regular where $\bigcup_{e \in E,\, f \in F} P_{ef} = \emptyset$.

Case 1b: Suppose aRa holds.

You do not have to complete this part of the proof.

Case 2: Consider the set of strings from a to b, where $a, b \in A$ and $a \neq b$.

You do not have to complete this part of the proof.