Automata Qual Exam (Fall 2014)
Answer ALL questions (Closed Book Exam)

1. (20 points)

Given a language $L \subseteq \Sigma^*$, and let $a \in \Sigma$, we define $\text{doubleFirst}_a(L) = \{waav \mid wav \in L,\, w \in (\Sigma - \{a\})^*,\, v \in \Sigma^*\}$.

We want to show that if L is regular, then $\text{doubleFirst}_a(L)$ is regular. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. You are asked to define formally another DFA M' in terms of Q, Σ, δ, q_0 and F such that $L(M') = \text{doubleFirst}_a(L(M))$. Your construction ideas should be clear. Otherwise, you need to supplement the formal construction with explanation.

Answer: Define $M' = (Q \times \{1, 2, 3\}, \Sigma, \delta', (q_0, 1), F \times \{3\})$ *as an incompletely specified DFA where* $\delta'((q, 1), x) = (\delta(q, x), 1)$ *for* $x \in \Sigma - \{a\}$, $\delta'((q, 1), a) = (q, 2)$, $\delta'((q, 2), a) = (\delta(q, a), 3)$, $\delta'((q, 3), x) = (\delta(q, x), 3)$ *for* $x \in \Sigma$.

2. (20 points)

Given a language $L \subseteq \Sigma^*$, and let $a \in \Sigma$, we define $\text{doubleSome}_a(L) = \{waav \mid wav \in L,\, w, v \in \Sigma^*\}$.

We want to show that if L is regular, then $\text{doubleSome}_a(L)$ is regular. To prove the result, you are asked to give a recursive algorithm that takes any regular expression r, and returns a regular expression r' such that $L(r') = \text{doubleSome}_a(L(r))$. Your construction ideas should be clear. Otherwise, you need to supplement the formal construction with explanation.

Answer:

```
f( r ) {
   case r of
      emptyset:         return emptyset
      emptystring:      return emptyset
      a:                return aa
      b:                return emptyset      // where b differs from a
      r1 U r2:          return f(r1) U f(r2)
      r1 r2:            return f(r1) r2   U   r1 f(r2)
      e*:               return r f(e) r
}
```

3. Let $A/B = \{w \mid wx \in A, \text{ for some } x \in B\}$.

(a) (20 points) If A is context-free and B is regular, is A/B necessarily context-free? Justify your answer.

Answer: Yes. Let M be a DFA for B and M' be a PDA for A. We construct a PDA for A/B by modifying M' so that it can run simultaneously (using the cartesian product) M's finite state control in addition to its own finite state control. Initially, only the original logic of M' is run. The modified machine, using nondeterminism, will start up M after w have been read. From this point onward, the machine guesses a string x, and processes x by both M and M'. The machine will accept only if both machines simultaneously accept.

(b) (20 points) If A is regular and B is context-free, is A/B necessarily regular? Justify your answer.

Answer: Yes. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for A. Given a state $q \in Q$, we define $L_q = \{x \mid \delta(q, x) \in F\}$. Then $M' = (Q, \Sigma, \delta, q_0, F')$ is a DFA for A/B where $F' = \{q \mid L_q \cap B \neq \emptyset\}$.

4.

Given a language L, we define a relation $\equiv_L$ on Σ^* such that $x \equiv_L y$ iff

$$\forall \alpha, \beta \in \Sigma^*,\ \alpha x \beta \in L \iff \alpha y \beta \in L$$

(a) (10 points) Prove that $\equiv_L$ is an equivalence relation.

Answer:

$\equiv_L$ is reflexive: Let $x \in \Sigma^*$. Since $\forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \iff \alpha x \beta \in L$, we have $x \equiv_L x$.

$\equiv_L$ is symmetric: Suppose $x \equiv_L y$. By definition, we have $\forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \iff \alpha y \beta \in L$. Equivalently, we have $\forall \alpha, \beta \in \Sigma^*, \alpha y \beta \in L \iff \alpha x \beta \in L$. Thus $y \equiv_L x$.

$\equiv_L$ is transitive: Suppose $x \equiv_L y$ and $y \equiv_L z$. By definition, we have $\forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \iff \alpha y \beta \in L$ and $\forall \alpha, \beta \in \Sigma^*, \alpha y \beta \in L \iff \alpha z \beta \in L$. Thus, $\forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \iff \alpha y \beta \in L \iff \alpha z \beta \in L$. Hence, $\forall \alpha, \beta \in \Sigma^*, \alpha x \beta \in L \iff \alpha z \beta \in L$. Therefore $x \equiv_L z$.

(b) (10 points) What are the equivalence classes of $\equiv_L$ for

$$L = \{a^h b^i c^j d^k \in \{a,b,c,d\}^* \mid h = k, \text{ or } i = j, \text{ where } h,i,j,k \geq 0\}$$

?

Answer: The set of equivalence classes is

$$
\begin{aligned}
&\{\{\epsilon\}\} \\
\cup&\{\{a^h\} \mid h > 0\} \\
\cup&\{\{b^i\} \mid i > 0\} \\
\cup&\{\{c^j\} \mid j > 0\} \\
\cup&\{\{d^k\} \mid k > 0\} \\
\cup&\{\{a^h b^i\} \mid h, i > 0\} \\
\cup&\{\{b^i c^j \mid i - j = p\} \mid p \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\} \\
\cup&\{\{c^j d^k\} \mid j, k > 0\} \\
\cup&\{\{a^h b^i c^j \mid i - j = p\} \mid h > 0,\ p \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\} \\
\cup&\{\{b^i c^j d^k \mid i - j = p\} \mid k > 0,\ p \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\} \\
\cup&\{\{a^h b^i c^i d^k \mid i \geq 0\} \mid h, k > 0\} \\
\cup&\{\{a^h b^i c^j d^k \mid i, j \geq 0,\ i \neq j, h - k = p\} \mid h, k > 0,\ p \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\} \\
\cup&\{\{x \in \{a,b,c,d\}^* \mid \forall \alpha, \beta \in \{a,b,c,d\}^*,\ \alpha x \beta \notin L\}\}
\end{aligned}
$$