An Introduction to Symbolic Logic

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1 Introduction

This project is dedicated to the study of basics of propositional and predicate logic. We will study it based on Russell and Whitehead's epoch making treatise *Principia Mathematica* [12].

Logic is a branch of science that studies correct forms of reasoning. It plays a fundamental role in such disciplines as philosophy, mathematics, and computer science. Like philosophy and mathematics, logic has ancient roots. The earliest treatises on the nature of correct reasoning were written over 2000 years ago. Some of the most prominent philosophers of ancient Greece wrote of the nature of deduction more than 2300 years ago, and thinkers in ancient China wrote of logical paradoxes around the same time [8]. However, though its roots may be in the distant past, logic continues to be a vibrant field of study to this day.

Modern logic originated in the work of the great Greek philosopher Aristotle (384–322 BC), the most famous student of Plato (c.427-c.347 BC) and one of the most influential thinkers of all time [11]. Further advances were made by the Greek Stoic philosopher Chrysippus of Soli (c.278–c.206 BC), who has developed the basics of what we now call propositional logic. More details on the work of Chrysippus can be found in [7].

For many centuries the study of logic was mostly concentrated on different interpretations of the works of Aristotle and to a much lesser degree of those of Chrysippus, whose work was largely forgotten [4]. However, the existing logic had no formal basis. All the argument forms were written in Greek,

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and lacked formal machinery that would create a logical calculus of deduction that is easy to work with.

The great German philosopher and mathematician Gottfried Willhelm Leibniz (1646–1716) was among te first to realize the need of formalizing logical argument forms. It was Leibniz's dream to create a universal formal language of science that would reduce all philosophical disputes to a matter of mere calculation by recasting the reasoning in such disputes in the universal symbolic language of science [7].

The first real steps in this direction were taken in the middle of the nineteenth century by the English mathematician George Boole (1815-1864). In 1854 Boole published *An Investigation of the Laws of Thought* [3], in which he developed an algebraic system for discussing logic. Boole's work ushered in a revolution in logic, which was advanced further by Augustus De Morgan (1806-1871), Charles Sanders Peirce (1839-1914), and Giuseppe Peano (1858-1932). Some of the work of Boole, De Morgan, and Peirce is discussed in [7] and [1]. Some of Peano's contributions are discussed in [2].

The next key step in this revolution in logic was made by the great German mathematician and philosopher Gottlob Frege (1848-1925). Frege created a powerful and profoundly original symbolic system of logic (Frege's formal treatment of propositional logic is discussed in [7]), as well as suggested that the whole of mathematics can be developed on the basis of formal logic, which resulted in the well-known school of *logicism*.

By the early twentieth century, the stage was set for Russell and Whitehead to give a modern account of logic in their influential treatise *Principia Mathematica*.

The son of a vicar in the Church of England [9, p. 23], Alfred North Whitehead (1861-1947) was born in Ramsgate, Kent, England, and studied mathematics at Trinity College. In 1884, Whitehead was elected a fellow at Trinity College, and would teach mathematics there until 1910. After his tenure at Trinity College, Whitehead spent time at University College London and Imperial College London, engaging in scholarly work in philosophy. He later emigrated to the United States, and taught philosophy at Harvard University until his retirement in 1937 [5].

While a fellow at Trinity College, Whitehead met Bertrand Russell (1872-1970), who was then a student at Trinity College [9, p. 223]. Russell was born into an aristocratic family. His grandfather, John Russell, 1st Earl Russell, was twice Prime Minister to Queen Victoria [10, p. 5]. Russell graduated from Trinity College in 1893. He went on to become one of the most influential intellectuals of the 20th century, playing a decisive role in the development of analytic philosophy. Russell was also active in a number of political causes (notably, he was an anti-war activist and advocated nuclear disarmament). He was a prolific writer, and received the Nobel Prize in Literature in 1950 [6].

Around 1901, Russell and Whitehead began collaborating on a book on logic and the foundations of mathematics [9, p. 254–258]. This resulted in an epochal work, *Principia Mathematica*, which would later be recognized as one of history's most significant contributions to logic and the foundations of mathematics.

In what follows, we will introduce the basic principles of contemporary logic through the development of Russell and Whitehead's *Principia Mathematica*.

2 Propositional Logic

In this section we begin our study of propositional logic from *Principia Mathematica*. The chief object of our investigation will be *propositions*—sentences which are either true or false but not both. Thus, we are concerned with sentences such as "Benjamin Franklin was the first president of the United States" and "two plus two is equal to four." Clearly the first of the two sentences is false and the second one is true. Therefore, both of the sentences are propositions. On the other hand, a sentence such as "who was the author of *Hamlet*?" is not a proposition because it is neither true nor false. Hence, we won't be concerned with these type of sentences.

To carry out our study of propositions, we introduce the concept of a *propositional variable*. Unlike the variables used in Algebra and Calculus, propositional variables do not merely stand for some undetermined quantity. Instead, propositional variables stand for propositions. The letters p, q, r and so forth will be used to denote propositional variables.

Logical connectives

We now turn to the first major topic in propositional logic, the question of how to form complicated propositions out of simpler ones. Russell and Whitehead address this question in the opening pages of *Principia Mathematica*:

> An aggregation of propositions...into a single proposition more complex than its constituents, is a function with *propositions as arguments.* [12, Vol. 1, p. 6]

Thus, more complex propositions are formed from simpler propositions by means of propositional functions. What are the propositional functions that yield more complex propositions out of simpler ones? Russell and Whitehead employ *four* fundamental propositional functions to build more complex propositions from simpler ones.

> ...They are (1) the Contradictory Function, (2) the Logical Sum, or Disjunctive Function, (3) the Logical Product, or Conjunctive Function, (4) the Implicative Function. These functions in the sense in which they are required in this work are not all independent; and if two of them are taken as primitive undefined ideas, the other two can be defined in terms of them. It is to some extent—though not entirely—arbitrary as to which functions are taken as primitive. Simplicity of primitive ideas and symmetry of treatment seem to be gained by taking the first two functions as primitive ideas. [12, Vol. 1, p. 6]

In the modern terminology the Contradictory Function of Russell and Whitehead is known as *negation* ("not"), the Logical Sum or Disjunctive Function is known as *disjunction* ("or"), the Logical Product or Conjunctive Function as *conjunction* ("and"), and the Implicative Function as *implication* ("if, then"). Statements of the form $p \supset q$ are referred to as *conditional statements*, or *conditionals*.

Russell and Whitehead mention that the four functions are not independent of each other. Later on we will see why this is so. For now, let us read how Russell and Whitehead define these four functions.

> The Contradictory Function with argument p, where p is any proposition, is the proposition which is the contradictory of p, that is, the proposition asserting that p is not true. This is denoted by $\sim p$. Thus $\sim p$ is the contradictory function with p as argument and means the negation of the proposition p. It will also be referred to as the proposition not-p. Thus $\sim p$ means not-p, which also means the negation of p.

The Logical Sum is a propositional function with two arguments p and q, and is the proposition asserting p or q disjunctively, that is, asserting that at least one of the two p and q is true. This is denoted by $p \vee q$. Thus $p \vee q$ is the logical sum with p and q as arguments. It is also called the logical sum of p and q. Accordingly $p \vee q$ means that at least p or q is true, not excluding the case in which both are true.

The Logical Product is a propositional function with two arguments p and q, and is the proposition asserting p and q conjunctively, that is, asserting that both p and q are true. This is denoted by $p.q$...Thus $p.q$ is the logical product with p and q as arguments. It is also called the logical product of p and q. Accordingly $p.q$ means that both p and q are true. It is easily seen that this function can be defined in terms of the two preceding functions. For when p and q are both true it must be false that either $\sim p$ or $\sim q$ is true. Hence in this book $p.q$ is merely a shortened form of symbolism for

$$\sim (\sim p \vee \sim q)$$

The Implicative Function is a propositional function with two arguments p and q, and is the proposition that either not-p or q is true, that is, it is the proposition $\sim p \vee q$. Thus if p is true, $\sim p$ is false, and accordingly the only alternative left by the proposition $\sim p \vee q$ is q is true. In other words if p and $\sim p \vee q$ are both true, then q is true. In this sense the proposition $\sim p \vee q$ will be quoted as stating that p implies q. The idea contained in this propositional function is so important that it requires a symbolism which with direct simplicity represents the propositions as connecting p and q without the intervention of $\sim p$. But "implies" as used here expresses nothing else than the connection between p and q also expressed by the disjunction "not-p or q." The symbol employed for "p implies q," *i.e.* for "$\sim p \vee q$," is "$p \supset q$." This symbol may also be read "if p, then q." [12, Vol. 1, p. 6–7]

Today we refer to the four fundamental functions of propositions of *Principia Mathematica* simply as *logical connectives*. Our first goal is to obtain a good understanding of propositions and of how the four logical connectives that

Russell and Whitehead introduced yield more complex propositions out of simpler ones.

Exercise 1. Which of the following sentences are propositions?

(a) The New York Yankees have never won a World Series.

(b) 2 is even.

(c) Please close the door.

(d) The square root of 109.

(e) The sum of two even integers is even.

(f) What is the capital of France?

For the sentences that are propositions, determine whether they are true or false.

Exercise 2. Let p denote the proposition "all mammals have four legs," q denote the proposition "all dogs have four legs," and r denote the proposition "all dogs are mammals." Represent each of the following propositions using the four fundamental functions of propositions of *Principia Mathematica*.

(a) Not all dogs have four legs.

(b) All mammals have four legs and all dogs have four legs.

(c) Not all mammals have four legs or not all dogs have four legs.

(d) If all mammals have four legs and all dogs are mammals, then all dogs have four legs.

(e) If not all dogs have four legs, then not all mammals have four legs or not all dogs have four legs.

Which of (a)—(e) are true and which are false?

Exercise 3. Let p denote the proposition "9 is odd," q denote the proposition "81 is the square of 9," and r denote the proposition "81 is odd." Write each of the following expressions in English.

(a) $p.q \supset r$

(b) $q.r \supset p$

(c) $\sim r \supset (\sim p \vee \sim q)$

(d) $p \vee \sim (q.r)$

Determine which of (a)—(d) are true and which are false.

Logical equivalence and biconditionals

In some cases, different propositions are, in some sense, logically the same. For example, the propositions "9 is odd and 81 is the square of 9" and "81 is the square of 9 and 9 is odd" are somehow alike despite having different symbolic representations. More generally, if p and q are propositions, the propositions $p.q$ and $q.p$ apparently have the same meaning. This property of being logically alike, called *logical equivalence*, is one of the most important concepts in propositional logic. *Principia Mathematica* describes logical equivalence as follows:

> The simplest example of the formation of a more complex function of propositions by the use of these four fundamental forms is furnished by "equivalence." Two propositions p and q are said to be "equivalent" when p implies q and q implies p. This relation between p and q is denoted by "$p \equiv q$." Thus "$p \equiv q$" stands for "$(p \supset q).(q \supset p)$." It is easily seen that two propositions are equivalent when, and only when, they are both true or are both false. [12, Vol. 1, p. 7]

Thus, two propositions are logically equivalent if they are true under the same conditions. Russell and Whitehead use special symbol $\equiv$ for denoting that p and q are logically equivalent. Since $p \equiv q$ stands for $(p \supset q).(q \supset p)$, $\equiv$ is also a logical connective, which can be expressed by means of . and $\supset$. Today this logical connective is known as the *biconditional*, and the statements of the form $p \equiv q$ are referred to as *biconditional statements*. We will see later that two propositions p and q are logically equivalent if and only if the biconditional $p \equiv q$ is always true.

It should be pointed out that although two propositions are logically equivalent whenever they are true under the same conditions, this does not mean that two logically equivalent propositions are alike in any other sense. *Principia Mathematica* provides a word of caution on this point:

> It must not be supposed that two propositions which are equivalent are in any sense identical or even remotely concerned with the same topic. Thus "Newton was a man" and "the sun is hot" are equivalent as being both true, and "Newton was not a man" and "the sun is cold" are equivalent as being both false. [12, Vol. 1, p. 8]

Truth-values

At first glance, it seems as if it may be difficult to determine whether two propositions are logically equivalent. For example, if p, q, and r are propositions, it is not readily apparent whether the compound propositions $(p \vee q).r$ and $(p.r) \vee (q.r)$ are logically equivalent. To develop a method for easily settling questions such as this, we first need to be able to determine whether a compound proposition is true or false. *Principia Mathematica* addresses this question:

> *Truth-values.* The "truth-value" of a proposition is *truth* if it is true, and *falsehood* if it is false. It will be observed that the truth-values of $p \vee q$, $p.q$, $p \supset q$, $\sim p$, $p \equiv q$ depend only on those of p and q, namely the truth-value of "$p \vee q$" is truth if the truth-value of either p or q is truth, and is falsehood otherwise; that of "$p.q$" is truth if that of both p and q is truth, and is falsehood otherwise; that of "$p \supset q$" is truth if either that of p is falsehood or that of q is truth; that of $\sim p$ is the opposite of that of p; and that of "$p \equiv q$" is truth if p and q have the same truth value, and is falsehood otherwise. [12, Vol. 1, p. 8]

In everyday English, propositions of the form "p or q" are usually intended to mean that either p is true or q is true, but not both. For example, in ordinary English the "or"-statement "the car was either red or blue" means that the car was red or blue, but not both red and blue. The logical connective "or" in such statements is called an *exclusive "or"*. However, according to *Principia Mathematica* the connective "$\vee$" expresses a different sort of "or"—for "$p \vee q$" is true if at *least one of* p and q is true. The connective $\vee$ is called an *inclusive "or"* because the truth of "$p \vee q$" allows for the possibility that both p and q are true. In mathematics and logic, "or"-statements always employ the inclusive "or" rather than the exclusive "or".

Note also that the truth-values of the implication "$p \supset q$" given in *Principia Mathematica* are somewhat different than one might expect. Russell and Whitehead indicate that "$p \supset q$" is true if either p is false or q is true; so, in particular, they regard statements of the form "if p, then q" as true if p is false. This way of thinking about "if, then" statements may at first seem unusual, but one may understand the motivation for doing so if one thinks of "$p \supset q$" as being analogous to a promise that if p holds true, then q will also hold true. If it so happens that p is not true, then the promise is unbroken and we regard "$p \supset q$" as true. The promise is broken only if p holds true, but q happens to be false. This is the only situation in which we will regard "$p \supset q$" as being false. Note that an implication "if p, then q" which is true because p is false is referred to as *vacuously true*.

Exercise 4. If the truth-value of p is truth and the truth-values of q and r are falsehood, compute the truth-values of the following compound propositions.

(a) $(p \vee q) \vee r$

(b) $p \supset (p.q)$

(c) $(\sim (p.r)) \supset \sim q$

(d) $(p \equiv r) \supset q$

Truth tables

We now understand how to determine the truth-value of a compound proposition from the truth-values of the propositional variables of which it is composed. However, to determine whether two compound propositions are logically equivalent we must be able to guarantee that they have the same truth-values under any circumstances. In other words, we must be able to guarantee that the compound propositions have the same truth-value for any possible assignment of truth-values to the propositional variables of which they are composed.

The great German philosopher Ludwig Wittgenstein (1889-1951) and the American logician Emil Post (1897-1954) independently devised a convenient method of listing the possible truth values of a compound proposition. This method employs *truth tables*, tables which list the possible truth-values of the propositional variables contained in a compound proposition along with the corresponding truth-values of the compound proposition itself. For example, if p is a propositional variable, then the truth table of the compound proposition $\sim p$ is as follows:

p	$\sim p$
T	F
F	T

Here the possible truth-values for the propositional variable p are listed in the column on the left, and the corresponding truth-values of the proposition $\sim p$ are listed in the column on the right. Truth is denoted by T and falsehood is denoted by F.

Compound propositions which contain more propositional variables have more complicated truth tables. For example, if p and q are propositional variables, then the truth table for the proposition $p \vee q$ is:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

It is now a simple matter to determine whether two compound propostions are logically equivalent: two propositions are logically equivalent if and only if they have the same truth tables. We can now answer the question of whether $(p \vee q).r$ and $(p.r) \vee (q.r)$ are logically equivalent by forming a truth table for these propositions:

p	q	r	$(p \vee q).r$	$(p.r) \vee (q.r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

As the two propositions have identical truth tables, they are logically equivalent.

Exercise 5. Construct truth tables for each of the following propositions.

(a) $p.q$

(b) $p \supset q$

(c) $p \equiv q$

(d) $(p \vee q) \supset r$

(e) $p \supset (q.r)$

Exercise 6. Use truth tables to show that:

(a) $\sim (p.q)$ is logically equivalent to $\sim p \vee \sim q$.

(b) $\sim (p \vee q)$ is logically equivalent to $\sim p. \sim q$

These equivalences are called *De Morgan's Laws* in honor of De Morgan.

Exercise 7. Are $p \supset q$ and $\sim q \supset \sim p$ logically equivalent? Justify your answer.

Now that we can easily construct truth tables of compound propositions, we will discuss Russell and Whiteheads claim (see page 4) that the four logical connectives are inter-definable. For example, Russell and Whitehead state that . and $\supset$ can be expressed by means of $\sim$ and $\vee$ as $\sim (\sim p \vee \sim q)$ and $\sim p \vee q$, respectively. In other words, $p.q$ is logically equivalent to $\sim (\sim p \vee \sim q)$ and $p \supset q$ is logically equivalent to $\sim p \vee q$.

Exercise 8.

(a) Show that $p.q$ is logically equivalent to $\sim (\sim p \vee \sim q)$.

(b) Show that $p \supset q$ is logically equivalent to $\sim p \vee q$.

Thus, . and $\supset$ are expressible by means of $\sim$ and $\vee$.

Exercise 9.

(a) Show that $\vee$ and $\supset$ are expressible by means of . and $\sim$.

(b) Show that $\vee$ and . are expressible by means of $\supset$ and $\sim$.

Tautologies

We have seen that truth tables are a useful tool for determining if two propositions are logically equivalent, but it turns out that truth tables have a variety of other uses. In what follows we will explore two other applications of truth tables. The first of these applications is the identification of certain special propositions. To illustrate what these propositions are and how truth tables can be used to identify them, we consider the truth table of the proposition $p \vee \sim p$:

p	$p \vee \sim p$
T	T
F	T

Notice that the truth-values displayed in the righthand column are always truth; falsity does not appear. Thus, $p \vee \sim p$ is true under any assignment of truth or falsity to the proposition p. Propositions with this property—that is, propositions which are true for any assignment of truth-values to the propositional variables of which they are formed—are called *tautologies*. From the truth table above, we can easily see that $p \vee \sim p$ is a tautology. $p \vee \sim p$ is referred to as *the law of the excluded middle* since it asserts that either p is true or the negation of p is true (so there is no "middle ground"). In general, one may test to see if a proposition is a tautology by constructing a truth table: if the only possible truth-value of the proposition is truth, then that proposition is a tautology.

Exercise 10. Use truth tables to verify that each of the following propositions is a tautology.

(a) $\sim (p. \sim p)$

(b) $p \equiv\sim (\sim p)$

(c) $(p \vee q) \supset (q \vee p)$

(d) $(p \supset q) \equiv (\sim q \supset\sim p)$

The logical laws expressed in (a) and (b) are called *the law of non-contradiction* and *the law of double negation*, respectively.

Now, as promised on page 7, we are ready to see that two propositions p and q are logically equivalent if and only if the biconditional $p \equiv q$ is a tautology.

Exercise 11. Show that p and q are logically equivalent if and only if $p \equiv q$ is a tautology.

Inference rules

Up until now we have been concerned exclusively with propositions and their properties. However, the central concern of logic is not just the study of propositions. Rather the object of study in logic is *inference*—the process of drawing correct conclusions from premises. Our study of inference begins with a simple rule which allows us to deduce the proposition q from the propositions p and $p \supset q$. This rule was known to the ancient Greeks as *Modus Ponens*. *Principia Mathematica* describes the rule as follows:

> *Inference*. The process of inference is as follows: a proposition "p" is asserted, and a proposition "p implies q" is asserted, and then as a sequel the proposition "q" is asserted. The trust in inference is the belief that if the two former assertions are not in error, the final assertion is not in error. [12, Vol. 1, p. 9]

Russell and Whitehead assert that Modus Ponens is a *valid* logical rule; that is, if the premises p and $p \supset q$ are both true, then the conclusion q is guaranteed to be true. While this fact is intuitively obvious, a skeptical reader may wonder how we really know this is the case. Fortunately, the skeptic's concerns may easily be allayed: we can verify that Modus Ponens preserves truth. For this we need to be able to guarantee the truth of q based on the truth of the propositions p and $p \supset q$. We thus need only to verify that q is true in every circumstance in which both p and $p \supset q$ are true. To do so, we construct a truth table listing the possible truth values of the propositions p, q, and $p \supset q$:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

The only row of the truth table in which both p and $p \supset q$ are true is the first. Since in this case q is also true, we know that if p and $p \supset q$ are both true, then q must be true as well. Thus, Modus Ponens is a valid logical rule.

So far we have focused exclusively on Modus Ponens, but it is worth noting that we might just as well have discussed any one of several valid logical rules. There are many valid logical rules (called *rules of inference*), and Modus Ponens is distinguished only by being the simplest of these. In the exercises, we will encounter several other historically important rules of inference and establish their validity.

Exercise 12. Use truth tables to verify the validity of the following rule of inference: if $\sim q$ and $p \supset q$ hold, then infer that $\sim p$ holds. (This rule of inference is referred to as *Modus Tollens*.)

Exercise 13. Use truth tables to verify the validity of the following rule of inference: if $\sim p$ and $p \vee q$ hold, then infer that q holds. (This rule of inference is referred to as the *disjunctive syllogism*.)

Exercise 14. Use truth tables to verify the validity of the following rule of inference: if $p \supset q$ and $q \supset r$ hold, then infer that $p \supset r$ holds. (This rule of inference is referred to as the *hypothetical syllogism*.)

3 Predicate Logic

While the propositional logic developed in the previous section allows us to address a number of significant issues in logic, it turns out that it is not capable of answering all of the logical questions which are important to mathematicians. For example, a mathematician may be interested in identifying the conditions under which the proposition "every prime number greater than 2 is odd" is true, but it is not obvious how to do this using propositional logic. It is apparently not possible to view this proposition as being composed of several elementary propositions joined by logical connectives, so from the perspective of propositional logic it must itself be an elementary proposition. However, if "every prime number greater than 2 is odd" is an elementary proposition, then propositional logic does not provide us with any information about when this proposition is true—the propositional logic developed in the previous section says only that elementary propositions are either true or false.

Individual variables

In order to provide a more informative analysis of the proposition "every prime number greater than 2 is odd," we must introduce ideas which allow us to discuss the internal structure of propositions that we previously regarded as elementary. In particular, in order to proceed we must introduce a new kind of variable. These variables, called *individual variables* to distinguish them from the propositional variables of the previous section, range over objects in some domain we are presently concerned with rather than over propositions. We will use the letters x, y, z and so forth to denote individual variables.

Predicates

With the notion of an individual variable at hand, we can now give an account of how "every prime number greater than 2 is odd" may be analyzed in terms of simpler propositions. Suppose that x is an individual variable, and that we take the expression $P(x)$ to mean that "x is a prime number greater than 2" and the expression $O(x)$ to mean that "x is odd." Then it is clear that the proposition "for every x, $P(x) \supset O(x)$" has the same meaning as "every prime number greater than 2 is odd." Thus, if we can formalize the meanings of $P(x)$, $O(x)$, and "for every x," then we can give an informative analysis of the proposition "every prime number greater than 2 is odd."

We will first formalize the meanings of the expressions $P(x)$ and $O(x)$. These expressions are examples of what Russell and Whitehead referred to as "propositional functions," which *Principia Mathematica* describes as follows:

> *Propositional Functions.* Let ϕx be a statement containing a variable x and such that it becomes a proposition when x is given any fixed determined meaning. Then ϕx is called a "propositional function"; it is not a proposition, since owing to the ambiguity of x it really makes no assertion at all. Thus "x is hurt" really makes no assertion at all, till we have settled who x is. Yet owing to the individuality retained by the ambiguous variable x, it is an ambiguous example from the collection of propositions arrived at by giving all possible determinations to x in "x is hurt" which yield a proposition, true or false. Also if "x is hurt" and "y is hurt" occur *in the same context*, where y is another variable, then according to the determinations given to x

> and y, they can be settled to be (possibly) the same propositions or (possibly) different propositions. But apart from some determination given to x and y, they retain in that context their ambiguous differentiation. Thus "x is hurt" is an ambiguous "value" of a propositional function. When we wish to speak of the propositional function corresponding to "x is hurt," we shall write "$\widehat{x}$ is hurt." Thus "$\widehat{x}$ is hurt" is the propositional function and "x is hurt" is an ambiguous value of that function. Accordingly through "x is hurt" and "y is hurt" *occurring in the same context* can be distinguished, "$\widehat{x}$ is hurt" and "$\widehat{y}$ is hurt" convey no distinction of meaning at all. More generally, ϕx is an ambiguous value of the propositional function $\phi\widehat{x}$, and when a definite signification a is substituted for x, ϕa is an unambiguous value of $\phi\widehat{x}$. [12, Vol. 1, p. 15]

As the statements "x is a prime number greater than 2" and "x is odd" become propositions when x is assigned a particular integer value, $P(x)$ and $O(x)$ as described above are propositional functions. Today the propositional functions of Russell and Whitehead are referred to as *predicates*.

If ϕ is a predicate, then the expression $\phi(x)$ may be thought of as asserting that x belongs to a particular collection of objects (namely, the collection of objects having whatever property the predicate attributes to x). If S is a collection, then $x \in S$ means that x is a member of the collection S.

It is common practice to denote the collection of natural numbers $(0, 1, 2$ and so forth) by $\mathbb{N}$, the collection of integers $(..., -2, -1, 0, 1, 2, ...)$ by $\mathbb{Z}$, the collection of rational numbers (ratios of two integers) by $\mathbb{Q}$, and the collection of real numbers by $\mathbb{R}$. Accordingly the predicates "x is a natural number," "x is an integer," "x is a rational number," and "x is a real number" will be expressed as "$x \in \mathbb{N}$," "$x \in \mathbb{Z}$," "$x \in \mathbb{Q}$," and "$x \in \mathbb{R}$," respectively.

Exercise 15. For the propositional functions $P(x)$ and $O(x)$ described above, state each of the following propositions in English. Also determine whether each proposition is true or false.

(a) $P(4) \vee O(7)$

(b) $(P(3).P(13)).O(12)$

(c) $P(2) \supset P(23)$

Universal and existential quantifiers

Now that we have introduced predicates, we are almost able to analyze the logic of the proposition "every prime number greater than 2 is odd." Recall that if we let $P(x)$ be the predicate "x is a prime number greater than 2" and $O(x)$ be the predicate "x is odd," then the proposition "every prime number greater than 2 is odd" can be expressed as "for every x, $P(x) \supset O(x)$." In order to fully understand the latter proposition, we only need to clarify what it means for a predicate $\phi(x)$ to hold "for every x." To do so, we will introduce the notion of a *quantifier*, a logical operator that specifies whether a predicate $\phi(x)$ holds for all values of x or some value of x. In *Principia Mathematica*, a predicate quantified in the former way is written as $(x).\phi(x)$ and a predicate quantified in the latter way is written as $(\exists x).\phi(x)$:

> *The range of values and total variation.* Thus corresponding to any propositional function $\phi\widehat{x}$, there is a range, or collection, of values, consisting of all the propositions (true or false) which can be obtained by giving every possible determination to x in ϕx. A value of x for which ϕx is true will be said to "satisfy" $\phi\widehat{x}$. Now in respect to the truth or falsehood of propositions of this range three important cases must be noted and symbolised. These cases are given by three propositions of which one at least must be true. Either (1) all propositions of the range are true, or (2) some propositions of the range are true, or (3) no proposition of the range is true. The statement (1) is symbolised by "$(x).\phi x$," and (2) is symbolised by "$(\exists x).\phi x$." No definition is given of these two symbols, which accordingly embody two new primitive ideas in our system. The symbol "$(x).\phi x$" may be read "ϕx always," or "ϕx is always true," or "ϕx is true for all possible values of x." The symbol "$(\exists x).\phi x$" may be read "there exists an x for which ϕx is true," or "there exists an x satisfying $\phi\widehat{x}$," and thus conforms to the natural form of the expression of thought. [12, Vol. 1, p. 15–16]

With quantifiers, we may now see that "every prime number greater than 2 is odd" may be represented symbolically as "$(x).(P(x) \supset O(x))$," where $P(x)$ is the predicate "x is a prime greater than 2" and $O(x)$ is the predicate "x is odd."

When we refer to propositions of the form $(x).\phi x$ and $(\exists x).\phi x$, we will always have in mind some fixed range of values over which x may vary. This range of values over which the relevant individual variables may range is called a *domain of discourse*, or simply a *domain*. Therefore, $(x).\phi(x)$ means that $\phi(x)$ holds for each value of the individual x in our domain of discourse. In other words, $(x).\phi(x)$ asserts that, for each value a in our domain of discourse, the proposition $\phi(a)$ holds. Thus, the proposition $(x).\phi(x)$ is true if, for each value a in our domain of discourse, the proposition $\phi(a)$ is true. On the other hand, $(x).\phi(x)$ is false if this condition fails to hold; that is, if for some value a in our domain of discourse, the proposition $\phi(a)$ is false.

Similarly, $(\exists x).\phi(x)$ means that there exists a value of x in our domain of discourse for which $\phi(x)$ holds. Thus, $(\exists x).\phi(x)$ asserts that there is some value a in our domain of discourse such that $\phi(a)$ holds. From this it is clear that $(\exists x).\phi(x)$ is true if there is some value a in our domain of discourse such that $\phi(a)$ holds, and that $(\exists x).\phi(x)$ is false if $\phi(a)$ happens to be false for every value a in our domain of discourse.

The symbol (x) is referred to as a *universal quantifier* since $(x).\phi x$ asserts that ϕx holds for all values of x, and the symbol $(\exists x)$ is referred to as an *existential quantifier* since $(\exists x).\phi x$ asserts that there exists a value of x for which ϕx holds.

Exercise 16. Let $E(x)$ be the predicate "x is an even number" and $O(x)$ be the predicate "x is an odd number." Express each of the following statements symbolically using quantifiers and the predicates $E(x)$ and $O(x)$.

(a) There is at least one even number.

(b) Every number is either even or odd.

(c) Some number is both even and odd.

(d) No number is both even and odd.

(e) Every number that is not even is odd.

(f) The square of every even number is even.

Which of the above statements are true and which are false?

Exercise 17. Let P denote the collection of prime numbers. Then $x \in P$ denotes the predicate "x is a prime number." Translate each of the following statements into English.

(a) $(x).(x \in \mathbb{N} \supset x \in \mathbb{Q})$

(b) $(\exists x).(x \in P \,.\, x \in \mathbb{R})$

(c) $(x).(x \in \mathbb{Z} \vee \sim (x \in \mathbb{Z}))$

Exercise 18. For each of the following statements, define appropriate predicates and write the statement using predicates and quantifiers.

(a) Every square is a rectangle.

(b) Every real number is either positive, negative, or equal to zero.

(c) Every animal that has a heart has kidneys, and every animal that has kidneys has a heart.

(d) Some natural number is not a prime number.

Unary and binary predicates

Our logical analysis of the proposition "every prime number greater than 2 is odd" relied on our ability to attribute certain properties to an individual variable x, and the introduction of predicates was what allowed us to accomplish this. Predicates are very versatile: they can provide symbolic representations for a multitude of interesting properties. For example, such properties as "being a rectangle," "being a prime number," "being greater than 2" are represented by the predicates "x is a rectangle," "x is a prime number," and "x is greater than 2," respectively. However, not every property that is of mathematical interest can be readily expressed using such predicates. For example, one may wish to express the fact that there exists a natural number n such that $n \leq m$ holds for all natural numbers m (i.e., the fact that the natural numbers have a least member).

The most obvious way to formalize this statement is to introduce a symbol $P(x, y)$ that expresses "x and y are natural numbers such that $x \leq y$." We may then express the existence of a least natural number by the proposition "$(\exists x).(y).P(x, y)$." In this expression, the role of the symbol $P(x, y)$ is similar to the role that predicates played in our previous discussion. Indeed, $P(x, y)$ is a sort of predicate, but, unlike the predicates previously discussed, $P(x, y)$ accepts two variables as arguments rather than one. Such a predicate is referred to as a *binary predicate*. Those predicates that attribute a property to only a single variable (such as those discussed previously) are referred to as *unary predicates.*

Exercise 19. Let $x \in \mathbb{N}$ denote the unary predicate "x is a natural number," $x \in \mathbb{Z}$ denote the unary predicate "x is an integer," and $P(x, y)$ denote the binary predicate "$x \leq y$." Represent each of the following propositions symbolically.

(a) There is a least natural number.

(b) There is a greatest natural number.

(c) There is a least integer.

(d) There is no greatest integer.

Which of the above propositions are true?

Exercise 20. Let $I(x, y)$ denote the binary predicate "the point x lies on the line y" (if a point x lies on a line y then x is said to be *incident* to y). Write each of the following propositions in English.

(a) $(x).(y).(\exists z).(I(x, z).I(y, z))$

(b) $(x).(\exists y).(I(y, x))$

What is the mathematical meaning of these two statements?

Exercise 21. Allowing the variables x and y to range over the domain of all lines in the Cartesian plane, let $P(x, y)$ denote the binary predicate "x is parallel to y." Write each of the following propositions in English.

(a) $(x).(P(x, x))$

(b) $(x).(y).(P(x, y) \supset P(y, x))$

(c) $(x).(y).(z).((P(x, y).P(y, z)) \supset P(x, z))$

When the three propositions above hold for a binary predicate $P(x, y)$, then the relationship defined between x and y by the predicate $P(x, y)$ is said to be an *equivalence relation.* Is "x is parallel to y" an equivalence relation?

Logical equivalence of quantified statements

Precisely characterizing when two quantified statements are logically equivalent turns out to be rather technical, and a thorough treatment of that topic is beyond the scope of this project. Nevertheless, from the comments above one can see that the proposition $(x).\phi(x)$ is true exactly when there is no value of x in our domain of discourse for which $\phi(x)$ is false. That is, $(x).\phi(x)$ is true exactly when $\sim (\exists x). \sim \phi(x)$ is true. One may likewise note that $(\exists x).\phi(x)$ is true exactly when $\phi(x)$ fails to be false for all x. In other words, $(\exists x).\phi(x)$ is true exactly when $\sim (x). \sim \phi(x)$ is true. As one may expect from this discussion, it so happens that $(x).\phi(x)$ is logically equivalent to $\sim (\exists x). \sim \phi(x)$ and that $(\exists x).\phi(x)$ is logically equivalent to $\sim (x). \sim \phi(x)$. One may derive many useful facts about the logical equivalence of quantified statements from these two laws.

Exercise 22. Use the laws

$$(x).\phi(x) \equiv \sim (\exists x). \sim \phi(x)$$

and

$$(\exists x).\phi(x) \equiv \sim (x). \sim \phi(x)$$

to rewrite each of the following statements.

(a) $\sim (\exists x).(y).(x < y)$

(b) $\sim (x).(\exists y).(x < y)$

Exercise 23. Allowing the variables x and y to range over the domain of all people, let $L(x, y)$ denote the binary predicate "x likes y." Write each of the following propositions symbolically.

(a) Everybody likes somebody.

(b) Somebody likes everybody.

Do the above propositions have the same meaning? What can you conclude about the relationship between universal quantifiers and existential quantifiers from the above example? In particular, are the two statements $(x).(\exists y).L(x, y)$ and $(\exists y).(x).L(x, y)$ logically equivalent? Justify your answer.

Inference rules in predicate logic

Our development of individual variables, predicates, and quantifiers has provided us with a language for discussing propositions that is much more expressive than the propositional language discussed in the previous section. In particular, the predicate logic is capable of addressing questions about complex propositions such as "every prime number greater than 2 is odd" that the propositional logic of the previous section cannot. However, we have not yet discussed the issue of logical inference in the predicate logic. This is the topic to which we now turn.

Recall that if p and q are propositional variables, then one can deduce the proposition q from the propositions p and $p \supset q$. We called this rule of inference Modus Ponens, and in the previous section we proved that Modus Ponens is a valid rule of inference (i.e., if p and $p \supset q$ are true, then the deduced proposition q must also be true). As it turns out, there is an analogous rule of inference for our predicate logic.

Suppose that we fix some domain over which our individuals variables may range, that a is some element of that domain, and that $P(x)$ and $Q(x)$ are predicates. Then if both $(x).(P(x) \supset Q(x))$ and $P(a)$ hold, we may deduce that $Q(a)$ holds as well. This gives a valid rule of inference, which we call *Universal Modus Ponens*.

A mathematically rigorous verification of the validity of Universal Modus Ponens requires an analogue of truth tables for predicate logic, and is beyond the scope of this project. However, we can give an informal argument for the validity of Universal Modus Ponens as follows.

Suppose that both $(x).(P(x) \supset Q(x))$ and $P(a)$ hold; we must show that $Q(a)$ holds as well. Since $(x).(P(x) \supset Q(x))$ holds, we have from the definition of the universal quantifier that the proposition $P(b) \supset Q(b)$ holds for each member b of our specified domain. In particular, as a is an element of the domain, we have that $P(a) \supset Q(a)$ holds. Now notice that both $P(a) \supset Q(a)$ and $P(a)$ are propositions. Since Modus Ponens is a valid rule of inference of propositional logic, we have that as $P(a) \supset Q(a)$ and $P(a)$ hold, it follows that $Q(a)$ holds as well, completing our argument.

Exercise 24. Fix some domain over which we allow individual variables to range, suppose a is an element of this domain, and further suppose that $P(x)$ and $Q(x)$ are predicates. Provide an informal verification that the following rule of inference is valid: If both $(x).(P(x) \supset Q(x))$ and $\sim Q(a)$ hold, infer that $\sim P(a)$ holds as well. (This rule is referred to as *Universal Modus Tollens*.)

4 Looking Forward

The logical system we have studied in this project is today known as *classical logic*. The publication of *Principia Mathematica* led to a considerable amount of research in classical logic in the first half of the 20th century, and the work of such logicians as Alfred Tarski (1901-1983) and Kurt Gödel (1906-1978) during this period has led to a thorough understanding of classical logic. However, original research in logic continues today. Contemporary research in logic focuses on a variety of so-called *non-classical logics*, each of which is in some way a modification or expansion of classical logic. The classical logic developed in this project hence forms the core of contemporary research, and thus remains of relevance almost a century after the publication of *Principia Mathematica*.

5 Note to the Instructor

This project has its roots in the authors' experience teaching discrete mathematics from primary historical sources at New Mexico State University (NMSU). It was designed to serve the needs of college sophomores who are meeting mathematical proofs for the first time. However, no prerequisites beyond elementary high school algebra are assumed.

The authors have attempted to include only those exercises which directly expand on the development of the material. As such, there are relatively few exercises, and almost all should be assigned to students.

Instructors should note that the notation introduced in *Principia Mathematica* is in some instances antiquated. In contemporary literature, the quantifier (x) is more commonly written as $\forall x$, and the symbols $\sim, \supset, \equiv$ have been largely supplanted by $\neg, \rightarrow$, and $\leftrightarrow$, respectively. Moreover, in *Principia Mathematica* propositional functions are denoted by juxtaposing the function name and its argument (e.g., $\phi\widehat{x}$). Instead of following this convention throughout the project, the authors have opted to enclose the argument of a propositional function in parentheses (e.g., $\phi(x)$), a notation that is more familiar to students at NMSU. For pedagogical reasons, they have also chosen to drop Russell and Whitehead's distinction between a propositional function $\phi\hat{x}$ and its ambiguous value ϕx.

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