Pick the Right Fence

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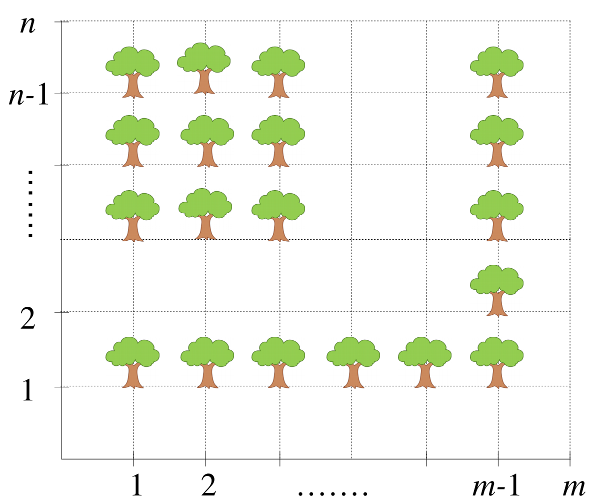
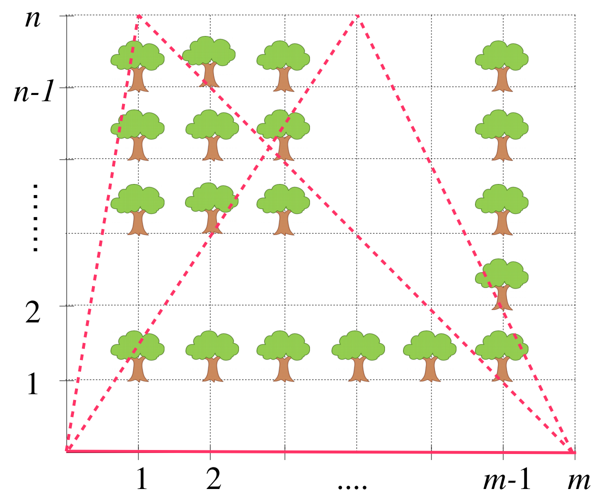
In a park there is an area including some rare trees. Such trees are planted very regularly at the points of a grid: each tree is determined by its integer coordinates (x, y) with $x \in [1, n-1], y \in [1, m-1]$, as in the figure below.



One day, the park is invaded by some mutant rabbits whose teeth are so strong that they can easily cut a tree trunk. The park is in danger and a fence should be installed to protect the trees. Unfortunately, the park authorities are facing some financial problems and they cannot afford the expense of a rectangular fence including the full grid. Only a triangular fence, with base on the x-axis, from $(0, 0)$ to $(m, 0)$ can be installed. The top vertex of the fence must be one of (x, n) for $x \in [1, m-1]$. Two possibilities are depicted by dotted lines in the figure below.



Please, help the park authorities to design properly the fence in a way that the largest number of trees will be protected. Remind that trees on the border of the fence will be necessarily cut in advance.

1. Show that, independently of the choice of top vertex (x, n), a triangular fence with no trees on the border will always include the same number, say i, of trees. Also observe that fences where some trees fall on the border will include a number of trees smaller than i.

   Hint: It is helpful to relate the area of a generic triangle with the number of trees it includes and the number of trees on its border.

2. Give a condition on the coordinates of the upper vertex of the triangle ensuring that no tree will be on the border.

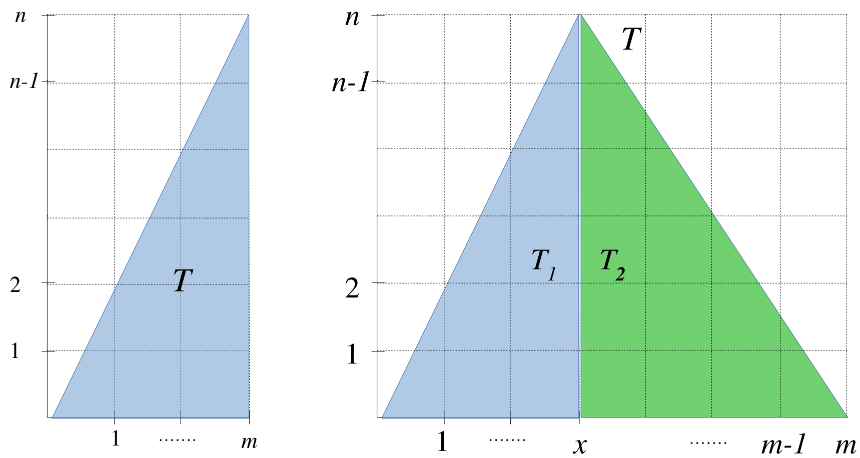
A solution.

1. A basic observation is that all triangles with the fixed base $(0, 0) - (0, m)$ and top vertex (x, n) have the same area $(mn)/2$. Moreover, the area $A(T)$ of a triangle T with integer coordinates is given by the formula

   $$A(T) = i + b/2 - 1 \tag{1}$$

   where i is the number of points with integer coordinates inside the triangle and b is the number of such points on the border of the triangle, including vertices. This is known, for general polygons, as Pick's theorem.

   A proof goes along the following lines. Consider a triangle consisting of half of a rectangle, like the blue one in the figure below, on the left (a):



(a) (b)

Its area is half of that of the rectangle, hence $A(T) = (mn)/2$. The number of boundary points is $b = m + n + 1 + h$ where h is the number of internal points intersected by the hypotenuse (in this case, $h = 2$). Since the total number of interior points for the rectangle is $(m-1)(n-1)$ of which h lie on the hypotenuse, the number of interior points for the triangle will be half of $(m-1)(n-1) - h$, i.e. $i = ((m-1)(n-1) - h)/2$. Therefore, we get that

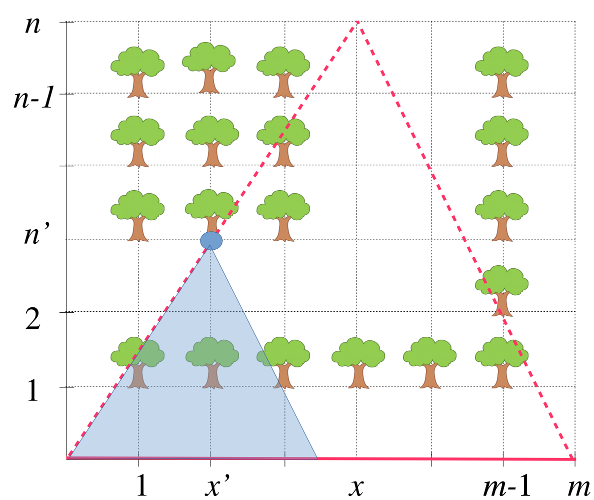
$$i + b/2 - 1 = \underbrace{((m-1)(n-1) - h)/2}_{i} + \underbrace{(m + n + 1 + h)/2}_{b/2} - 1 = (mn)/2$$

which is correctly the area of the triangle. Now, observe that a triangle T of the shape of the fence, as the figure above (b), can be divided in two rectangle triangles T_1 and T_2, to which the previous proof applies. We thus just need to note that the formula (1) is additive. In fact, for triangles T_1 and T_2 we have $A(T_1) = i_1 + b_1/2$ and $A(T_2) = i_2 + b_2/2$, where i_j and b_j denotes the number of internal and border points for the two subtriangles. Note that T_1 and T_2 have exactly $n+1$ points in common on their borders, of which $n-1$ are internal to T and 2 are on the border of T. If i and b are the numbers of internal and border points for the full triangle T it clearly holds $i = i_1 + i_2 + n - 1$ and $b = (b_1 - (n+1)) + (b_2 - (n+1)) + 2 = b_1 + b_2 - 2n$. Therefore:

$$
\begin{aligned}
i + b/2 - 1 &= i_1 + i_2 + n - 1 + (b_1 + b_2 - 2n)/2 - 1 \\
&= i_1 + i_2 - 1 + (b_1 + b_2)/2 - 1 \\
&= (i_1 + b_1/2 - 1) + (i_2 + b_2/2 - 1) \\
&= A(T_1) + A(T_2) \\
&= A(T)
\end{aligned}
$$

Once Pick's formula (1) is established, both assertions in point 1 trivially follows.

2. A necessary and sufficient condition to avoid trees (points with integer coordinates) on the border is to choose x in a way that x and n are *coprime*, i.e., their greatest common divisor is 1. In fact, assume that there is an integer point on the border, and call (x', n') its coordinates, with $x' < x$ as in the picture below.



Since the large triangle and the small (blue) one are similar, we have that $xn' = x'n$. Take any prime p and $k \geq 0$ such that: i) p^{k+1} divides x and not x', and ii) p^k divides both x and x' (there must be one such prime as otherwise, since each number have a prime factorization, we would have that x divides x' whence $x \leq x'$, contradicting the hypothesis). Obviously p^{k+1} divides xn' and thus $x'n$. Therefore p divides n, and x, n are not coprime. It is apparent that also the converse holds, i.e., if x and n are coprime then no integer point can be on the border.

Oserve that, in particular, one can always choose $x = 1$, which clearly ensures that x and n are coprime.