Representing Actions in Extended Logic Programming

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Abstract

We represent properties of actions in a logic programming language that uses both classical negation and negation as failure. The method is applicable to temporal projection problems with incomplete information, as well as to reasoning about the past. It is proved to be sound relative to a semantics of action based on states and transition functions.

1 Introduction

This paper extends the work of Eshghi and Kowalski [4], Evans [6] and Apt and Bezem [1] on representing properties of actions in logic programming languages with negation as failure.

Our goal is to overcome some of the limitations of the earlier work. The existing formalizations of action in logic programming are adequate for only the simplest kind of temporal reasoning—"temporal projection." In a temporal projection problem, we are given a description of the initial state of the world, and use properties of actions to determine what the world will look like after a series of actions is performed. Moreover, the existing formalizations can be used for temporal projection only in the cases when the given description of the initial state is complete. The reason for that is that traditional logic programming languages automatically apply the "closed world assumption" to each predicate.
We are interested here in temporal reasoning of a more general kind, when the values of some fluents in one or more situations are given, and the goal is to derive other facts about the values of fluents. Besides temporal projection, this class of reasoning problems includes, for instance, the cases when we want to use information about the current state of the world for answering questions about the past. The view of logic programming accepted in this paper is strictly declarative. The adequacy of a representation of a body of knowledge in a logic programming language means, to us, adequacy with respect to the declarative semantics of that language. It is interesting to find out, of course, whether any of the currently available query evaluation procedures will actually terminate if used for answering questions on the basis of our representation, and how fast, but those are secondary issues. In fact, the language of "extended logic programs" used in this paper is a subset of the system of default logic from [16], and our work can be viewed as a development of the approach to temporal reasoning based on nonnormal defaults [15].

Two parts of this paper may be of more general interest.

First, we introduce here a simple declarative language for describing actions, called $\mathcal{A}$. Traditionally, ideas on representing properties of actions in classical logic or nonmonotonic formalisms are explained on specific examples, such as the "Yale shooting problem" from [9]. Competing approaches are evaluated and compared in terms of their ability to handle such examples. Sandewall [17] provides a systematic comparison of the most important approaches by applying them to a rather long series of problems of this kind. We propose to supplement the use of examples by a different method. A particular methodology for representing action can be formally described as a translation from $\mathcal{A}$, or from a subset or a superset of $\mathcal{A}$, into a "target language"—for instance, into a language based on classical logic or on circumscription. The soundness and completeness of each particular translation become precise mathematical questions; the possibilities and limitations of each methodology can be described in terms of the "dialects" of $\mathcal{A}$ to which it is applicable. Our method for describing properties of actions in logic programming is presented here as a translation from $\mathcal{A}$ into the language of extended logic programs, and the soundness of this translation is the main technical result of the paper.

Second, the proof of the main theorem depends on a relationship between stable models [7] and signings [11], that may be interesting as a part of the general theory of logic programming.

The language $\mathcal{A}$ is introduced in Section 2, and Section 3 is a brief review of extended logic programs. Our translation from $\mathcal{A}$ into logic programming is defined in Section 4, and the soundness theorem is stated in Section 5. Section 6 contains the lemmas that relate stable models to signings. The proof of the soundness theorem can be found in the complete version of this paper.
A description of an action domain in the language \( A \) consists of "propositions" of two kinds. A "v-proposition" specifies the value of a fluent in a particular situation—either in the initial situation, or after performing a sequence of actions. An "e-proposition" describes the effect of an action on a fluent.

We begin with two disjoint nonempty sets of symbols, called fluent names and action names. A fluent expression is a fluent name possibly preceded by \( \neg \). A v-proposition is an expression of the form

\[
F \text{ after } A_1; \ldots; A_m,
\]

where \( F \) is a fluent expression, and \( A_1; \ldots; A_m \) \((m \geq 0)\) are action names. If \( m = 0 \), we will write (1) as

initially \( F \).

An e-proposition is an expression of the form

\[
A \text{ causes } F \text{ if } F_1; \ldots; P_n,
\]

where \( A \) is an action name, and each of \( F, F_1; \ldots; P_n \) \((n \geq 0)\) is a fluent expression. About this proposition we say that it describes the effect of \( A \) on \( F \), and that \( F_1; \ldots; P_n \) are its preconditions. If \( n = 0 \), we will drop if and write simply

\[
A \text{ causes } F.
\]

A proposition is a v-proposition or an e-proposition. A domain description, or simply domain, is a set of propositions (not necessarily finite).

**Example 1.** The Fragile Object domain, motivated by an example from [18], has the fluent names Holding, Fragile and Broken, and the action Drop. It consists of two e-propositions:

\[
\begin{align*}
\text{Drop causes } \neg \text{Holding if Holding}, \\
\text{Drop causes Broken if Holding, Fragile.}
\end{align*}
\]

**Example 2.** The Yale Shooting domain, motivated by the example from [9] mentioned above, is defined as follows. The fluent names are Loaded and Alive; the action names are Load, Shoot and Wait. The domain is characterized by the propositions

\[
\begin{align*}
\text{initially } \neg \text{Loaded}, \\
\text{initially Alive}, \\
\text{Load causes Loaded}, \\
\text{Shoot causes } \neg \text{Alive if Loaded}, \\
\text{Shoot causes } \neg \text{Loaded}.
\end{align*}
\]
Example 3. The Murder Mystery domain, motivated by an example from [2], is obtained from the Yale Shooting domain by substituting

\[ \neg \text{Alive after Shoot; Wait} \]  

for the proposition initially \( \neg \text{Loaded} \).

Example 4. The Stolen Car domain, motivated by an example from [10], has one fluent name \textit{Stolen} and one action name \textit{Wait}, and is characterized by two propositions:

initially \( \neg \text{Stolen} \),
\[ \text{Stolen after Wait; Wait; Wait}. \]

To describe the semantics of \( A \), we will define what the “models” of a domain description are, and when a v-proposition is “entailed” by a domain description.

A state is a set of fluent names. Given a fluent name \( F \) and a state \( \sigma \), we say that \( F \) holds in \( \sigma \) if \( F \in \sigma \); \( \neg F \) holds in \( \sigma \) if \( F \notin \sigma \). A transition function is a mapping \( \Phi \) of the set of pairs \( (A, \sigma) \), where \( A \) is an action name and \( \sigma \) is a state, into the set of states. A structure is a pair \( (\sigma_0, \Phi) \), where \( \sigma_0 \) is a state (the initial state of the structure), and \( \Phi \) is a transition function.

For any structure \( M \) and any action names \( A_1, \ldots, A_m \), by \( M^{A_1:\ldots:A_m} \) we denote the state

\[ \Phi(A_m, \Phi(A_{m-1}, \ldots, \Phi(A_1, \sigma_0) \ldots)), \]

where \( \Phi \) is the transition function of \( M \), and \( \sigma_0 \) is the initial state of \( M \). We say that a v-proposition (1) is true in a structure \( M \) if \( F \) holds in the state \( M^{A_1:\ldots:A_m} \), and that it is false otherwise. In particular, a proposition of the form initially \( F \) is true in \( M \) iff \( F \) holds in the initial state of \( M \).

A structure \( (\sigma_0, \Phi) \) is a model of a domain description \( D \) if every v-proposition from \( D \) is true in \( (\sigma_0, \Phi) \), and, for every action name \( A \), every fluent name \( F \), and every state \( \sigma \), the following conditions are satisfied:

(i) if \( D \) includes an e-proposition describing the effect of \( A \) on \( F \) whose preconditions hold in \( \sigma \), then \( F \in \Phi(A, \sigma) \);

(ii) if \( D \) includes an e-proposition describing the effect of \( A \) on \( \neg F \) whose preconditions hold in \( \sigma \), then \( F \notin \Phi(A, \sigma) \);

(iii) if \( D \) does not include such e-propositions, then \( F \in \Phi(A, \sigma) \) iff \( F \in \sigma \).

It is clear that there can be at most one transition function \( \Phi \) satisfying conditions (i)–(iii). Consequently, different models of the same domain description can differ only by their initial states. For instance, the Fragile Object domain (Example 1) has 8 models, whose initial states are the subsets of

\{\textit{Holding}, \textit{Fragile}, \textit{Broken}\};
in each model, the transition function is defined by the equation

\[ \Phi(Drop, \sigma) = \begin{cases} \sigma \setminus \{Holding\} \cup \{Broken\}, & \text{if } \text{Fragile} \in \sigma, \\ \sigma \setminus \{Holding\}, & \text{otherwise.} \end{cases} \]

A domain description is consistent if it has a model, and complete if it has exactly one model. The Fragile Object domain is consistent, but incomplete. The Yale Shooting domain (Example 2) is complete; its only model is defined by the equations:

\[ \sigma_0 = \{Alive\}, \]
\[ \Phi(\text{Load}, \sigma) = \sigma \cup \{\text{Loaded}\}, \]
\[ \Phi(\text{Shoot}, \sigma) = \begin{cases} \sigma \setminus \{\text{Loaded}, \text{Alive}\}, & \text{if } \text{Loaded} \in \sigma, \\ \sigma, & \text{otherwise,} \end{cases} \]
\[ \Phi(\text{Wait}, \sigma) = \sigma. \]

The Murder Mystery domain (Example 3) is complete also; it has the same transition function as Yale Shooting, and the initial state \( \{\text{Loaded}, \text{Alive}\} \). The Stolen Car domain (Example 4) is inconsistent.

A \( v \)-proposition is entailed by a domain description \( D \) if it is true in every model of \( D \). For instance, Yale Shooting entails

\( \neg \text{Alive after Load; Wait; Shoot.} \)

Murder Mystery entails, among others, the propositions

initially \( \text{Loaded} \)

and

\( \neg \text{Alive after Wait; Shoot.} \)

Note that the last proposition differs from (3) by the order in which the two actions are executed. This example illustrates the possibility of reasoning about alternative “possible futures” of the initial situation.

Although the language \( A \) is adequate for formalizing several interesting domains, its expressive possibilities are rather limited. The only fluents available in \( A \) are propositional ones. It is impossible to say in \( A \) that certain fluents are related in some way (for instance, that \( F1 \) and \( F2 \) cannot be simultaneously true); for this reason, actions described in \( A \) have no indirect effects (“ramifications”). Every action is assumed to be executable in any situation. We cannot talk about the duration of actions, or describe actions that are nondeterministic, or are performed concurrently. The inconsistency of the Stolen Car domain illustrates the fact that \( A \) cannot be used for representing “causal anomalies,” or “miracles” [12]². Defining and studying extensions of \( A \) is a topic for future work.

The entailment relation of \( A \) is nonmonotonic, in the sense that adding an \( e \)-proposition to a domain description \( D \) may nonmonotonically change the set of propositions entailed by \( D \). (This cannot happen when a \( v \)-proposition
is added.) For this reason, a modular translation from a declarative language (that is, a translation that processes propositions one by one) can be reasonably adequate only if this other language is nonmonotonic also.

3 Extended Logic Programs

Representing incomplete information in traditional logic programming languages is difficult, because their semantics is based on the automatic application of the closed world assumption to all predicates. Given a ground query, a traditional logic programming system can produce only one of two answers, yes or no; it will never tell us that the truth value of the query cannot be determined on the basis of the information included in the program.

Extended logic programs, introduced in [8], are, in this sense, different. The language of extended programs distinguishes between negation as failure not and classical negation \( \neg \). The general form of an extended rule is

\[
L_0 \leftarrow L_1, \ldots, L_m, \neg L_{m+1}, \ldots, \neg L_n,
\]

where each \( L_i \) is a literal, that is, an atom possibly preceded by \( \neg \). An extended program is a set of such rules. Here is an example:

\[
p, \\
\neg q \leftarrow p, \\
r \leftarrow \neg p, \\
t \leftarrow \neg q, \neg s, \\
u \leftarrow \neg \neg u.
\]

Intuitively, these rules say:

- \( p \) is true;
- \( q \) is false if \( p \) is true;
- \( r \) is true if \( p \) is false;
- \( t \) is true if \( q \) is false and there is no evidence that \( s \) is true;
- \( u \) is true if there is no evidence that it is false.

The answers that an implementation of this language is supposed to give to the ground queries are:

\[
p : \text{yes}, \\
q : \text{no}, \\
r : \text{unknown}, \\
s : \text{unknown}, \\
t : \text{yes}, \\
u : \text{yes}.
\]
The semantics of extended logic programs defines when a set of ground literals is an answer set of a program \([8]\). For instance, the program (5) has one answer set, \(\{ \neg q, t, u \}\).

The answer sets of a program can be easily characterized in terms of default logic. We will identify the rule (4) with the default:

\[
L_1 \land \ldots \land L_m : \overline{L_{m+1}}, \ldots, \overline{L_n} / L_0
\]

(\(\overline{L}\) stands for the literal complementary to \(L\)). Thus every extended program can be viewed as a default theory. The answer sets of a program are simply its extensions in the sense of default logic, intersected with the set of ground literals (\([8]\), Proposition 3).

4 Describing Actions by Logic Programs

Now we are ready to define the translation \(\pi\) from \(A\) into the language of extended programs.

About two different e-propositions we say that they are similar if they differ only by their preconditions. Our translation method is defined for any domain description that does not contain similar e-propositions. This condition prohibits, for instance, combining in the same domain such propositions as

\[
\text{Shoot causes } \neg \text{Alive if Loaded},
\]

\[
\text{Shoot causes } \neg \text{Alive if VeryNervous}.
\]

(\text{VeryNervous} refers to the victim, of course—not to the gun.)

Let \(D\) be a domain description without similar e-propositions. The corresponding logic program \(\pi D\) uses variables of three sorts: situation variables \(s, s', \ldots\), fluent variables \(f, f', \ldots\), and action variables \(a, a', \ldots\). Its only situation constant is \(S0\); its fluent constants and action constants are, respectively, the fluent names and action names of \(D\). There are also some predicate and function symbols; the sorts of their arguments and values will be clear from their use in the rules below.

The program \(\pi D\) will consist of the translations of the individual propositions from \(D\) and the four standard rules:

\[
\text{Holds}(f, \text{Result}(a, s)) \leftarrow \text{Holds}(f, s), \neg \text{Noninertial}(f, a, s),
\]

\[
\neg \text{Holds}(f, \text{Result}(a, s)) \leftarrow \neg \text{Holds}(f, s), \neg \text{Noninertial}(f, a, s),
\]

\[
\text{Holds}(f, s) \leftarrow \text{Holds}(f, \text{Result}(a, s)), \neg \text{Noninertial}(f, a, s),
\]

\[
\neg \text{Holds}(f, s) \leftarrow \neg \text{Holds}(f, \text{Result}(a, s)), \neg \text{Noninertial}(f, a, s).
\]

These rules are motivated by the “commonsense law of inertia,” according to which the value of a fluent after performing an action is normally the same as before. The rules (7) allow us to apply the law of inertia in reasoning “from the past to the future”: the first—when a fluent is known to be true
in the past, and the second—when it is known to be false. The rules (8)
play the same role for reasoning "from the future to the past." The auxiliary
predicate Noninertial is essentially an "abnormality predicate" [13].

Now we will define how \( \pi \) translates v-propositions and e-propositions.
The following notation will be useful: For any fluent name \( F \),

\[
| F \rangle \text{ is } F, \quad | \neg F \rangle \text{ is } F,
\]

and, if \( t \) is a situation term, \( \text{Holds}(\neg F, t) \) stands for \( \neg \text{Holds}(F, t) \). The last
convention allows us to write \( \text{Holds}(F, t) \) even when \( F \) is a fluent name
preceded by \( \neg \). Furthermore, if \( A_1, \ldots, A_m \) are action names, \( [A_1; \ldots; A_m] \)
stands for the term

\[
\text{Result}(A_m, \text{Result}(A_{m-1}, \ldots, \text{Result}(A_1, S_0), \ldots)).
\]

It is clear that every situation term without variables can be represented in
this form.

The translation of a v-proposition (1) is

\[
\text{Holds}(F, [A_1; \ldots; A_m]). \quad (9)
\]

For instance, \( \pi \) (initially Alive) is

\[
\text{Holds}(\text{Alive}, S_0),
\]

and \( \pi \) (Alive after Shoot) is

\[
\neg \text{Holds}(\text{Alive}, \text{Result}(\text{Shoot}, S_0)).
\]

The translation of an e-proposition (2) consists of \( 2n + 2 \) rules. The first
of them is

\[
\text{Holds}(F, \text{Result}(A, s)) \leftarrow \text{Holds}(P_1, s), \ldots, \text{Holds}(P_n, s). \quad (10)
\]

It allows us to prove that \( F \) will hold after \( A \), if the preconditions are
satisfied. The second rule is

\[
\text{Noninertial}(|F|, A, s) \leftarrow \overline{\text{Holds}(P_1, s)} \ldots, \overline{\text{Holds}(P_n, s)} \quad (11)
\]

(\( \text{Holds}(P_i, s) \) is the literal complementary to \( \text{Holds}(P_i, s) \).) It disables the
inertia rules (7), (8) in the cases when \( f \) can be affected by \( a \). Without this
rule, the program would be contradictory: We would prove, using a rule of
the form (10), that an unloaded gun becomes loaded after the action Load,
and also, using the second of the rules (7), that it remains unloaded!

Note the use of \( \overline{\text{not}} \) in (11). We want to disable the inertia rules not
only when the preconditions for the change in the value of \( F \) are known
to hold, but whenever there is no evidence that they do not hold. If, for
instance, we do not know whether Loaded currently holds, then we do not
want to conclude by inertia that the value of Alive will remain the same after Shoot. We cannot draw any conclusions about the new value of Alive. If we replaced the body of (11) by \( \text{Holds}(P_1, s), \ldots, \text{Holds}(P_n, s) \), the translation would become unsound.

Besides (10) and (11), the translation of (2) contains, for each \( 1 \leq i \leq n \), the rules

\[
\text{Holds}(P_i, s) \leftarrow \text{Holds}(F, s), \text{Holds}(F, \text{Result}(A, s))
\]

and

\[
\begin{align*}
\text{Holds}(P_i, s) & \leftarrow \text{Holds}(F, \text{Result}(A, s)), \\
& \quad \text{Holds}(P_2, s), \ldots, \text{Holds}(P_{i-1}, s), \\
& \quad \text{Holds}(P_{i+1}, s), \ldots, \text{Holds}(P_n, s).
\end{align*}
\]

The rules (12) justify the following form of reasoning: If the value of \( F \) has changed after performing \( A \), then we can conclude that the preconditions were satisfied when \( A \) was performed. These rules would be unsound in the presence of similar propositions. The rules (13) allow us to conclude that a precondition was false from the fact that performing an action did not lead to the result described by an effect axiom, while all other preconditions were true.

We will illustrate the translation process by applying it to Yale Shooting (Example 2). The translation of that domain includes, in addition to (7) and (8), the following rules:

\[
\begin{align*}
Y1. \quad & \neg \text{Holds(Loaded, S0)}. \\
Y2. \quad & \text{Holds(Alive, S0)}. \\
Y3. \quad & \text{Holds(Loaded, Result(Load, s))}. \\
Y4. \quad & \text{Noninertial(Loaded, Load, s)}.
\end{align*}
\]

\[
\begin{align*}
Y5. \quad & \neg \text{Holds(Alive, Result(Shoot, s))} \leftarrow \text{Holds(Loaded, s)}. \\
Y6. \quad & \text{Noninertial(Alive, Shoot, s)} \leftarrow \neg \text{Holds(Loaded, s)}. \\
Y7. \quad & \text{Holds(Loaded, s)} \leftarrow \text{Holds(Alive, s)}, \neg \text{Holds(Alive, Result(Shoot, s))}. \\
Y8. \quad & \neg \text{Holds(Loaded, s)} \leftarrow \text{Holds(Alive, Result(Shoot, s))}. \\
Y9. \quad & \neg \text{Holds(Loaded, Result(Shoot, s))}. \\
Y10. \quad & \text{Noninertial(Loaded, Shoot, s)}.
\end{align*}
\]

It is instructive to compare this set of rules with the formalization of Yale Shooting given by Apt and Bezem [1], who were only interested in temporal projection problems, and did not use classical negation. Instead of our four inertia rules, they have one, corresponding to the first of the rules (7). In addition, their program includes counterparts of \( Y2, Y3, Y5 \) and \( Y6 \). It does not tell us whether \( \text{Loaded} \) holds in the initial situation, but the negative answer to this question follows by the closed world assumption. Their rule
corresponding to Y5 does not have ¬ in the head, of course; instead, the
new fluent Dead is used. In their counterpart of Y6, the combination not ¬
is missing; this does not lead to any difficulties, because the closed world
assumption is implicitly postulated.

5 Soundness Theorem

We say that a ground literal L is entailed by an extended logic program, if it belongs to all its answer sets (or, equivalently, to all its extensions in the
sense of default logic). Using this notion of entailment and the entailment
relation for the language A introduced in Section 2, we can state a result
expressing the soundness of the translation π.

Soundness Theorem. Let D be a domain description without similar e-
propositions. For any v-proposition P, if πD entails πP, then D entails
P.

For an inconsistent D, the statement of the soundness theorem is
trivial, because such D entails every v-proposition. For consistent domain
descriptions, the statement of the theorem is an immediate consequence of
the following lemma which will be proved elsewhere:

Soundness Lemma. Let D be a consistent domain description without
similar e-propositions. There exists an answer set Z of πD such that, for
any v-proposition P, if πP ∈ Z then D entails P.

Note that the lemma asserts the possibility of selecting Z uniformly for
all P; this is more than is required for the soundness theorem.

The set Z from the statement of the lemma is obviously consistent,
because a consistent domain description cannot entail two complementary
v-propositions. Consequently, if D is consistent and does not include similar
v-propositions, then πD has a consistent answer set.

The converse of the soundness theorem does not hold, so that the
translation π is incomplete. This following simple counterexample belongs
to Thomas Woos (personal communication). Let D be the domain with one
fluent name F and one action name A, characterized by two propositions:

\[
A \text{ after } F,
\]
\[
A \text{ causes } F \text{ if } F.
\]

It is clear that D entails initially F. But the translation of this proposition,
\(\text{Holds}(F, S0)\), is not entailed by πD. Indeed, it is easy to verify that the set
of all positive ground literals other than \(\text{Holds}(F, S0)\) is an answer set of
πD.
6 Answer Sets and Signings

To prove the soundness lemma, we need the following definition. Let $\Pi$ be a general logic program (that is, an extended program that does not contain classical negation). A signing for $\Pi$ is any set $S$ of ground atoms such that, for any ground instance

$$B_0 \leftarrow B_1, \ldots, B_m, \neg B_{m+1}, \ldots, \neg B_n$$

of any rule from $\Pi$, either

$$B_0, B_1, \ldots, B_m \in S, B_{m+1}, \ldots, B_n \not\in S$$

or

$$B_0, B_1, \ldots, B_m \not\in S, B_{m+1}, \ldots, B_n \in S.$$

For example, \{p\} is a signing for the program

$$p \leftarrow \neg q, \quad q \leftarrow \neg p, \quad r \leftarrow q.$$

In this section we show that the answer sets of a general program $\Pi$ which has a signing $S$ can be characterized in terms of the fixpoints of a monotone operator. Specifically, for any set $X$ of ground atoms, let $\theta X$ be the symmetric difference of $X$ and $S$:

$$\theta X = (X \setminus S) \cup (S \setminus X).$$

Obviously, $\theta$ is one to one. Moreover, it is clear that $\theta$ is an involution:

$$\theta^2 X = \{(X \setminus S) \cup (S \setminus X) \setminus S \cup (S \setminus X)\} \cup \{S \setminus (S \setminus X)\}$$

$$= (X \setminus S) \cup (S \setminus X) = X.$$

We will define a monotone operator $\phi$ such that any $X$ is an answer set of $\Pi$ if and only if $\theta X$ is a fixpoint of $\phi$.

Recall that, for general logic programs, the notion of an answer set (or "stable model") can be defined by means of the following construction [7]. Let $\Pi$ be a general logic program, with every rule replaced by all its ground instances. The reduct $\Pi^X$ of $\Pi$ relative to a set $X$ of ground atoms is obtained from $\Pi$ by deleting

(i) each rule that has an expression of the form $\neg B$ in its body with $B \in X$, and

(ii) all expressions of the form $\neg B$ in the bodies of the remaining rules.

Clearly, $\Pi^X$ is a positive program, and we can consider its "minimal model"—the smallest set of ground atoms closed under its rules. If this set coincides with $X$, then $X$ is an answer set of $\Pi$. 
This condition can be expressed by the equation $X = \alpha \Pi X$, where $\alpha$ is the operator that maps any positive program to its minimal model.

Let $S$ be a signing for $\Pi$. The operator $\phi$ is defined by the equation

$$\phi X = \theta \alpha \Pi \phi X.$$

**Lemma 1.** A set $X$ of ground atoms is an answer set of $\Pi$ iff $\theta X$ is a fixpoint of $\phi$.

**Proof.** By the definition of $\phi$, $\theta X$ is a fixpoint of $\phi$ iff

$$\theta \alpha \Pi \phi X = \theta X.$$

Since $\theta$ is one-to-one and an involution, this is equivalent to

$$\alpha \Pi X = X.$$

Note that, since $\theta$ is an involution, Lemma 1 can be also stated as follows: $X$ is an answer set of $\Pi$ iff $X = \theta Y$ for some fixpoint $Y$ of $\phi$.

**Lemma 2.** The operator $\phi$ is monotone.

**Proof.** Let $\Pi_1$ be the set of all rules from $\Pi$ whose heads belong to $S$, and let $\Pi_2$ be the set of all remaining rules. Clearly, for any $X$,

$$\Pi X = \Pi_1 X \cup \Pi_2 X.$$

Since $S$ is a signing for $\Pi$, all atoms occurring in $\Pi_1 X$ belong to $S$, and all atoms occurring in $\Pi_2 X$ belong to the complement of $S$. Consequently, $\Pi_1 X$ and $\Pi_2 X$ are disjoint, and

$$\alpha \Pi X = \alpha \Pi_1 X \cup \alpha \Pi_2 X.$$

Furthermore, for any expression of the form $\text{not } B$ occurring in $\Pi_1$, $B$ does not belong to $S$; consequently,

$$\Pi_1 X = \Pi_1 X \setminus S.$$

Similarly, for any expression of the form $\text{not } B$ occurring in $\Pi_2$, $B$ belongs to $S$, so that

$$\Pi_2 X = \Pi_2 X \cap S.$$

Consequently, for every $X$,

$$\alpha \Pi X = \alpha \Pi_1 X \setminus S \cup \alpha \Pi_2 X \cap S.$$

In particular,

$$\alpha \Pi \phi X = \alpha \Pi_1 \phi X \setminus S \cup \alpha \Pi_2 \phi X \cap S.$$
It is clear from the definition of $\theta$ that

$$\theta X \setminus S = X \setminus S,$$

$$\theta X \cap S = S \setminus X.$$  

We conclude that

$$\alpha \Pi^X \setminus S = \alpha \Pi^S \setminus X.$$

By the choice of $\Pi_1$ and $\Pi_2$, $\alpha \Pi^X \setminus S$ is contained in $S$, and $\alpha \Pi^S \setminus X$ is disjoint with $S$. Consequently,

$$\alpha \Pi^X \setminus S = \alpha \Pi^S \setminus X,$$

$$S \setminus \alpha \Pi^X = S \setminus \alpha \Pi^S \setminus X.$$ 

Hence

$$\phi X = \theta \alpha \Pi^X = (\alpha \Pi^X \setminus S) \cup (S \setminus \alpha \Pi^X) = \alpha \Pi^S \setminus X \cup (S \setminus \alpha \Pi^X \setminus S).$$

Since $\alpha$ is monotone, and the reduct operators $X \mapsto \Pi^X$ are antimonotone, it follows that $\phi$ is monotone.

Having proved Lemmas 1 and 2, we can use properties of the fixpoints of monotone operators given by the Knaster-Tarski theorem [20] to study the answer sets of a program with a signing. The Knaster-Tarski theorem asserts, for instance, that every monotone operator has a fixpoint; this gives a new, and more direct, proof of the fact that every general program with a signing has at least one answer set. Moreover, it asserts that a monotone operator has a least fixpoint, which is also its least pre-fixpoint. (A pre-fixpoint of $\phi$ is any set $X$ such that $\phi X \subseteq X$.) This characterization of the least fixpoint of $\phi$ is used in the proof of the soundness lemma.

Footnotes

1. A fluent is something that may depend on the situation, as, for instance, the location of a moveable object [14]. In particular, propositional fluents are assertions that can be true or false depending on the situation.

2. One possible way to represent reasoning about the past in the framework of logic programming is to interpret it as "explanation" and "abduction" [19]. Our approach is more symmetric; we treat both reasoning about the future and reasoning about the past as "deductive." The precise relationship between the two approaches is a subject of further investigation.

3. Our preferred approach to causal anomalies is to view them as evidence of unknown events that occur concurrently with the given actions and contribute to the properties of the new situation.

4. Using a sorted language implies, first of all, that all atoms in the rules of the program are formed in accordance with the syntax of sorted predicate
logic. Moreover, when we speak of an instance of a rule, it will be always assumed that the terms substituted for variables are of appropriate sorts.

5. This is slightly different from the original definition [11].

6. The existence of answer sets for such programs, and for programs of some more general types, was established by Phan Minh Dung [3] and François Fages [6].

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References


