EXAMPLES OF FUNCTIONS

There are numerous ways to write the square function:

1. As an *extensional* (infinite) set of pairs: $\{[1.2,1.44],[3.7,13.69],[0.0,0.0],\ldots\}$
2. As an *intensional* set of pairs: $\{[x,y] \mid \forall x \in \mathbb{R}, y = x^2\}$
3. As a relation: $\forall [x,y] \in \mathbb{R} \times \mathbb{R}, x \text{ square } y \Leftrightarrow y = x^2$
4. As a typed lambda expression: $\lambda x \in \mathbb{R} . x^2$
5. Functionality: $\text{square}: \mathbb{R} \to \mathbb{R}$
6. Formula: $\text{square}: x \to x^2$
7. "standard" math form: $square(x) = x^2$, for all x

If we have the functions $\text{square}: x \to x^2$ and $\text{successor}: x \to x+1$, then the composition $\text{square} \circ \text{successor}$ is the function $\text{squareSuccessor}: x \to (x+1)^2$, since $\text{successor}(x)$ is $x+1$ and $\text{square}(x+1)$ is $(x+1)^2$.

Functions with more than one argument can be written as, e.g.:

1. $\lambda x, y . x + 2y$
2. $f(x,y) = x + 2y$
3. $\{[[x,y],z] \mid \forall z, y, z \in \mathbb{R}, z = x + 2y\}$

although it can be shown that a function of more than one argument can be "curried" (after mathematician Haskell Curry) into a composition of single argument functions:

$\lambda x, y . x + 2y$ is the same as $\lambda x . \lambda y . x + 2y$.

The application of a function can be achieved by textual substitution of the argument into the body of the function. Whether to evaluate the argument first is a matter of choice, but the most common way is to completely evaluate the argument before substitution. E.g.

(standard notation) $square(square(3+1)) = square(square(4)) = square(16) = 256$

(lambda notation) $(\lambda x.x^2)(\lambda x.x^2(1+3)) = (\lambda x.x^2)(\lambda x.x^2(4)) = \lambda x.x^2(16) = 256$

[Notice how the lambda form has everything right there, including the body of the function, whereas the standard form has to "remember" the definition body.]

An updated function can also be applied: $([4 \mapsto 1]\{\})4 = 1$, but $([4 \mapsto 1]\{\})3$ is undefined, since the function does not map 3 to anything.