Algorithms and Data Structures CS 372

Recurrences

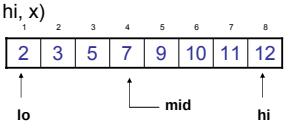
(Based on slides by M. Nicolescu)

Recursive Algorithms BINARY-SEARCH

- for an ordered array A, finds if x is in the array A[lo…hi]

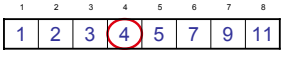
Alg.: BINARY-SEARCH (A, lo, hi, x)

```
if (lo > hi)
    return FALSE
mid ← ⌊(lo+hi)/2⌋
if x = A[mid]
    return TRUE
if ( x < A[mid] )
    BINARY-SEARCH (A, lo, mid-1, x)
if ( x > A[mid] )
    BINARY-SEARCH (A, mid+1, hi, x)
```

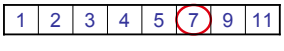


Example

- A[8] = {1, 2, 3, 4, 5, 7, 9, 11}
  - lo = 1 hi = 8 x = 7



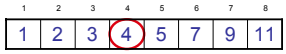
mid = 4, lo = 5, hi = 8



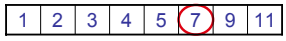
mid = 6, A[mid] = x
Found!

Example

- A[8] = {1, 2, 3, 4, 5, 7, 9, 11}
  - lo = 1 hi = 8 x = 6



mid = 4, lo = 5, hi = 8



mid = 6, A[6] = 7, lo = 5, hi = 5

mid = 5, A[5] = 5, lo = 6, hi = 5
NOT FOUND!

4

Analysis of BINARY-SEARCH

Alg.: BINARY-SEARCH (A, lo, hi, x)

```
if (lo > hi)                                   ← constant time: c1
    return FALSE
mid ← ⌊(lo+hi)/2⌋                              ← constant time: c2
if x = A[mid]                                  ← constant time: c3
    return TRUE
if ( x < A[mid] )
    BINARY-SEARCH (A, lo, mid-1, x)            ← same problem of size n/2
if ( x > A[mid] )
    BINARY-SEARCH (A, mid+1, hi, x)            ← same problem of size n/2
```

- $T(n) = c + T(n/2)$
  - T(n) – running time for an array of size n

5

Recurrences and Running Time

- Recurrences arise when an algorithm contains recursive calls to itself
- What is the actual running time of the algorithm?
- Need to solve the recurrence
  - Find an explicit formula of the expression (the generic term of the sequence)

6

Example Recurrences

- $T(n) = T(n-1) + n$ $\Theta(n^2)$
  - Recursive algorithm that loops through the input to eliminate one item
- $T(n) = T(n/2) + c$ $\Theta(\lg n)$
  - Recursive algorithm that halves the input in one step
- $T(n) = T(n/2) + n$ $\Theta(n)$
  - Recursive algorithm that halves the input but must examine every item in the input
- $T(n) = 2T(n/2) + 1$ $\Theta(n)$
  - Recursive algorithm that splits the input into 2 halves and does a constant amount of other work

7

Methods for Solving Recurrences

- Iteration method
- Substitution method
- Recursion tree method
- Master method

8

The Iteration Method

$T(n) = c + T(n/2)$

$T(n) = c + T(n/2)$ $\quad T(n/2) = c + T(n/4)$

$= c + c + T(n/4)$ $\quad T(n/4) = c + T(n/8)$

$= c + c + c + T(n/8)$

Assume $n = 2^k$

$T(n) = \underbrace{c + c + \ldots + c}_{k \text{ times}} + T(1)$

$= c\lg n + T(1)$

$= \Theta(\lg n)$

9

Iteration Method – Example

$T(n) = n + 2T(n/2)$ Assume: $n = 2^k$

$T(n) = n + 2T(n/2)$ $T(n/2) = n/2 + 2T(n/4)$

$= n + 2(n/2 + 2T(n/4))$

$= n + n + 4T(n/4)$

$= n + n + 4(n/4 + 2T(n/8))$

$= n + n + n + 8T(n/8)$

... $= in + 2^iT(n/2^i)$

$= kn + 2^kT(1)$

$= nlgn + nT(1) = \Theta(nlgn)$

10

The substitution method

1. Guess a solution
2. Use induction to prove that the solution works

11

Substitution method

- Guess a solution
  - $T(n) = O(g(n))$
  - Induction goal: apply the definition of the asymptotic notation
    - $T(n) \le d\,g(n)$, for some $d > 0$ and $n \ge n_0$
  - Induction hypothesis: $T(k) \le d\,g(k)$ for all $k < n$
- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

12

Example: Binary Search

$T(n) = c + T(n/2)$

- Guess: $T(n) = O(\lg n)$
  - Induction goal: $T(n) \le d \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le d \lg(n/2)$
- Proof of induction goal:

$T(n) = T(n/2) + c \le d \lg(n/2) + c$

$= d \lg n - d + c \le d \lg n$

if: $-d + c \le 0, d \ge c$

13

Example 2

$T(n) = T(n-1) + n$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le c n^2$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n-1) \le c(n-1)^2$ for all $k < n$
- Proof of induction goal:

$T(n) = T(n-1) + n \le c (n-1)^2 + n$

$= cn^2 - (2cn - c - n) \le cn^2$

if: $2cn - c - n \ge 0 \Leftrightarrow c \ge n/(2n-1) \Leftrightarrow c \ge 1/(2 - 1/n)$

  - For $n \ge 1 \Rightarrow 2 - 1/n \ge 1 \Rightarrow$ any $c \ge 1$ will work

14

Example 3

$T(n) = 2T(n/2) + n$

- Guess: $T(n) = O(n \lg n)$
  - Induction goal: $T(n) \le cn \lg n$, for some c and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le cn \lg(n/2)$
- Proof of induction goal:

$T(n) = T(n/2) + n \le 2c (n/2)\lg(n/2) + n$

$= cn \lg n - cn + n \le cn \lg n$

if: $-cn + n \le 0 \Rightarrow c \ge 1$

15

Changing variables

$T(n) = 2T(\sqrt{n}) + \lg n$

– Rename: $m = \lg n \Rightarrow n = 2^m$

$T(2^m) = 2T(2^{m/2}) + m$

– Rename: $S(m) = T(2^m)$

$S(m) = 2S(m/2) + m \Rightarrow S(m) = O(m \lg m)$
(demonstrated before)

$T(n) = T(2^m) = S(m) = O(m \lg m) = O(\lg n \lg \lg n)$

Idea: transform the recurrence to one that you have seen before

16

Changing variables (cont.)

$T(n) = 2T(n/2) + 1$

– Rename: $n = 2^m$ $(n/2 = 2^{m-1})$

$T(2^m) = 2T(2^{m-1}) + 1$

– Rename: $S(m) = T(2^m)$

$S(m) = S(m-1) + 1$

$= 1 + S(m-1) = 1 + 1 + S(m-2) = \underbrace{1 + 1 + \ldots + 1}_{m-1 \text{ times}} + S(1)$

$S(m) = O(m) \Rightarrow T(n) = T(2^m) = S(m) = O(m) = O(\lg n)$

17

The recursion-tree method

Convert the recurrence into a tree:

– Each node represents the cost incurred at various levels of recursion

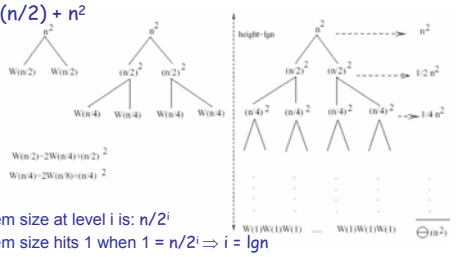
– Sum up the costs of all levels

Used to "guess" a solution for the recurrence

18

Example 1

$W(n) = 2W(n/2) + n^2$



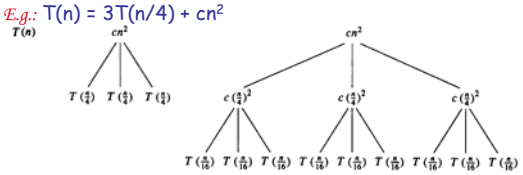
- Subproblem size at level i is: $n/2^i$
- Subproblem size hits 1 when $1 = n/2^i \Rightarrow i = \lg n$
- Cost of the problem at level $i = (n/2^i)^2$ No. of nodes at level $i = 2^i$
- Total cost:

$$W(n) = \sum_{i=0}^{\lg n-1} \frac{n^2}{2^i} + 2^{\lg n} W(1) = n^2 \sum_{i=0}^{\lg n-1} \left(\frac{1}{2}\right)^i + n \le n^2 \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i + O(n) = n^2 \frac{1}{1-1/2} + O(n) = 2n^2$$

$\Rightarrow W(n) = O(n^2)$

Example 2

E.g.: $T(n) = 3T(n/4) + cn^2$



- Subproblem size at level i is: $n/4^i$
- Subproblem size hits 1 when $1 = n/4^i \Rightarrow i = \log_4 n$
- Cost of a node at level $i = c(n/4^i)^2$
- Number of nodes at level $i = 3^i \Rightarrow$ last level has $3^{\log_4 n} = n^{\log_4 3}$ nodes
- Total cost:

$$T(n) = \sum_{i=0}^{\log_4 n-1} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) \le \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) = \frac{1}{1-\frac{3}{16}} cn^2 + \Theta\left(n^{\log_4 3}\right) = O(n^2)$$

$\Rightarrow T(n) = O(n^2)$

Example 2 - Substitution

$T(n) = 3T(n/4) + cn^2$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le dn^2$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/4) \le d\,(n/4)^2$
- Proof of induction goal:

$T(n) = 3T(n/4) + cn^2$

$\le 3d\,(n/4)^2 + cn^2$

$= (3/16)\,d\,n^2 + cn^2$

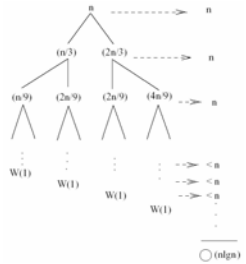
$\le d\,n^2$ if: $d \ge (16/13)c$

- Therefore: $T(n) = O(n^2)$

Example 3

$W(n) = W(n/3) + W(2n/3) + O(n)$

- The longest path from the root to a leaf is:
  $n \to (2/3)n \to (2/3)^2 n \to \ldots \to 1$
- Subproblem size hits 1 when
  $1 = (2/3)^i n \Leftrightarrow i = \log_{3/2} n$
- Cost of the problem at level i = n
- Total cost:



$$\underbrace{n + n + \ldots}_{\text{For all levels}} < \sum_{i=0}^{\log_{3/2} n} n = n \sum_{i=0}^{\log_{3/2} n} 1 = n \log_{3/2} n = n \frac{\lg n}{\lg 3/2} = \frac{1}{\lg 3/2} n \lg n$$

$\Rightarrow W(n) = O(n \lg n)$

Example 3 - Substitution

$W(n) = W(n/3) + W(2n/3) + O(n)$

- Guess: $W(n) = O(n \lg n)$
  - Induction goal: $W(n) \le dn \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $W(k) \le d\, k \lg k$ for any $K < n$ (n/3, 2n/3)
- Proof of induction goal:

  Try it out as an exercise!!
- $T(n) = O(n \lg n)$

Master's method

- "Cookbook" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \ge 1$, $b > 1$, and $f(n) > 0$

Case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

Case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$af(n/b) \le cf(n)$ for some $c < 1$ and all sufficiently large n, then:

regularity condition

$T(n) = \Theta(f(n))$

Why $n^{\log_b a}$?

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T(n) = aT\left(\frac{n}{b}\right)$$
$$a^2T\left(\frac{n}{b^2}\right)$$
$$a^3T\left(\frac{n}{b^3}\right)$$
$$\vdots$$
$$T(n) = a^iT\left(\frac{n}{b^i}\right) \quad \forall i$$

- Assume $n = b^k \Rightarrow k = \log_b n$
- At the end of iteration i = k:

$$T(n) = a^{\log_b n}T\left(\frac{b^i}{b^i}\right) = a^{\log_b n}T(1) = \Theta\left(a^{\log_b n}\right) = \Theta\left(n^{\log_b a}\right)$$

- Case 1:
  - If $f(n)$ is dominated by $n^{\log_b a}$:
    - $T(n) = \Theta(n^{\log_b a})$
- Case 3:
  - If $f(n)$ dominates $n^{\log_b a}$:
    - $T(n) = \Theta(f(n))$
- Case 2:
  - If $f(n) = \Theta(n^{\log_b a})$:
    - $T(n) = \Theta(n^{\log_b a} \log n)$

25

Examples

$T(n) = 2T(n/2) + n$

$a = 2, b = 2, \log_2 2 = 1$

Compare $n^{\log_2 2}$ with $f(n) = n$

$\Rightarrow f(n) = \Theta(n) \Rightarrow$ Case 2

$\Rightarrow T(n) = \Theta(n \lg n)$

26

Examples

$T(n) = 2T(n/2) + n^2$

$a = 2, b = 2, \log_2 2 = 1$

Compare n with $f(n) = n^2$

$\Rightarrow f(n) = \Omega(n^{1+\varepsilon})$ Case 3 $\Rightarrow$ verify regularity cond.

$a\, f(n/b) \le c\, f(n)$

$\Leftrightarrow 2\, n^2/4 \le c\, n^2 \Rightarrow c = \frac{1}{2}$ is a solution ($c<1$)

$\Rightarrow T(n) = \Theta(n^2)$

27

Examples (cont.)

$T(n) = 2T(n/2) + \sqrt{n}$

$a = 2, b = 2, \log_2 2 = 1$

Compare n with $f(n) = n^{1/2}$

$\Rightarrow f(n) = O(n^{1-\varepsilon})$ Case 1

$\Rightarrow T(n) = \Theta(n)$

28

Examples

$T(n) = 3T(n/4) + n\lg n$

$a = 3, b = 4, \log_4 3 = 0.793$

Compare $n^{0.793}$ with $f(n) = n\lg n$

$f(n) = \Omega(n^{\log_4 3+\varepsilon})$ Case 3

Check regularity condition:

$3*(n/4)\lg(n/4) \le (3/4)n\lg n = c*f(n), c=3/4$

$\Rightarrow T(n) = \Theta(n\lg n)$

29

Examples

$T(n) = 2T(n/2) + n\lg n$

$a = 2, b = 2, \log_2 2 = 1$

- Compare n with $f(n) = n\lg n$
  - seems like case 3 should apply
- f(n) must be polynomially larger by a factor of n^{ε}
- In this case it is only larger by a factor of lg

30