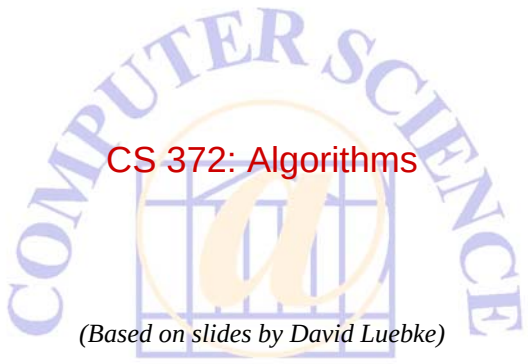


CS 372: Algorithms

(Based on slides by David Luebke)

Review: Induction

- Suppose
  - S(k) is true for fixed constant k
    - Often k = 0
  - S(n) → S(n+1) for all n >= k
- Then S(n) is true for all n >= k

Proof By Induction

- Claim:S(n) is true for all n >= k
- Basis:
  - Show formula is true when n = k
- Inductive hypothesis:
  - Assume formula is true for an arbitrary n
- Step:
  - Show that formula is then true for n+1

Induction Example: Gaussian Closed Form

- Prove $1 + 2 + 3 + \ldots + n = n(n+1) / 2$
  - Basis:
    - If $n = 0$, then $0 = 0(0+1) / 2$
  - Inductive hypothesis:
    - Assume $1 + 2 + 3 + \ldots + n = n(n+1) / 2$
  - Step (show true for n+1):

    $1 + 2 + \ldots + n + n+1 = (1 + 2 + \ldots + n) + (n+1)$

    $= n(n+1)/2 + n+1 = [n(n+1) + 2(n+1)]/2$

    $= (n+1)(n+2)/2 = (n+1)(n+1 + 1) / 2$

Induction Example: Geometric Closed Form

- Prove $a^0 + a^1 + \ldots + a^n = (a^{n+1} - 1)/(a - 1)$ for all $a \neq 1$
  - Basis: show that $a^0 = (a^{0+1} - 1)/(a - 1)$

    $a^0 = 1 = (a^1 - 1)/(a - 1)$
  - Inductive hypothesis:
    - Assume $a^0 + a^1 + \ldots + a^n = (a^{n+1} - 1)/(a - 1)$
  - Step (show true for n+1):

    $a^0 + a^1 + \ldots + a^{n+1} = a^0 + a^1 + \ldots + a^n + a^{n+1}$

    $= (a^{n+1} - 1)/(a - 1) + a^{n+1} = (a^{n+1+1} - 1)/(a - 1)$

Induction

- We've been using *weak induction*
- *Strong induction* also holds
  - Basis: show S(0)
  - Hypothesis: assume S(k) holds for arbitrary k <= n
  - Step: Show S(n+1) follows
- Another variation:
  - Basis: show S(0), S(1)
  - Hypothesis: assume S(n) and S(n+1) are true
  - Step: show S(n+2) follows

Analysis of Algorithms

- Analysis is performed with respect to a computational model
- We will usually use a generic uniprocessor random-access machine (RAM)
  - All memory equally expensive to access
  - No concurrent operations
  - All reasonable instructions take constant time
    - Except, of course, function calls
  - Limit on word size
    - Input of size n is represented by c lg n bits

Input Size

- Time and space complexity
  - This is generally a function of the input size
    - E.g., sorting, multiplication
  - How we characterize input size depends:
    - Sorting: number of input items
    - Multiplication: total number of bits
    - Graph algorithms: number of nodes & edges
    - Etc

Running Time

- Number of primitive steps that are executed
  - Except for time of executing a function call most statements roughly require the same amount of time
    - y = m * x + b
    - c = 5 / 9 * (t - 32)
- We can be more exact if need be

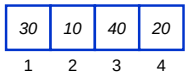
Analysis

- Worst case
  Provides an upper bound on running time
  An absolute guarantee
- Average case
  Provides the expected running time
- Best case

An Example: Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

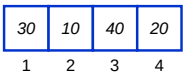
An Example: Insertion Sort



i = ∅ j = ∅ key = ∅
A[j] = ∅ A[j+1] = ∅

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

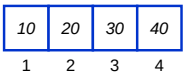
An Example: Insertion Sort



i = 2 j = 1 key = 10
A[j] = 30 A[j+1] = 10

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

An Example: Insertion Sort



i = 4 j = 1 key = 20
A[j] = 10 A[j+1] = 20

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
            A[j+1] = A[j]
            j = j - 1
      }
      A[j+1] = key
  }
}
```

Done!

Insertion Sort

Statement	Effort
`InsertionSort(A, n) {`	
`  for i = 2 to n {`	$c_1 n$
`    key = A[i]`	$c_2(n-1)$
`    j = i - 1;`	$c_3(n-1)$
`    while (j > 0) and (A[j] > key) {`	$c_4 T$
`      A[j+1] = A[j]`	$c_5(T-(n-1))$
`      j = j - 1`	$c_6(T-(n-1))$
`    }`	0
`    A[j+1] = key`	$c_7(n-1)$
`  }`	0
`}`	

$T = t_2 + t_3 + \ldots + t_n$ where t_i is number of while expression evaluations for the i^{th} for loop iteration

Analyzing Insertion Sort

- $T(n) = c_1n + c_2(n-1) + c_3(n-1) + c_4T + c_5(T - (n-1)) + c_6(T - (n-1)) + c_7(n-1)$
  $= c_8T + c_9n + c_{10}$
- What can T be?
  - Best case -- inner loop body never executed
    - $t_i = 1$ ➔ T(n) is a linear function
  - Worst case -- inner loop body executed for all previous elements
    - $t_i = i$ ➔ T(n) is a quadratic function
  - Average case
    - ???

Upper Bound Notation

- We say InsertionSort's run time is $O(n^2)$
  - Properly we should say run time is *in* $O(n^2)$
  - Read O as "Big-O" (you'll also hear it as "order")
- In general a function
  - f(n) is O(g(n)) if there exist positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$
- Formally
  - O(g(n)) = { f(n): ∃ positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ ∀ $n \geq n_0$ }

Insertion Sort Is $O(n^2)$

- Proof
  - Suppose runtime is $an^2 + bn + c$
    - If any of a, b, and c are less than 0 replace the constant with its absolute value
  - $an^2 + bn + c \leq (a + b + c)n^2 + (a + b + c)n + (a + b + c)$
  - $\leq 3(a + b + c)n^2$ for $n \geq 1$
  - Let $c' = 3(a + b + c)$ and let $n_0 = 1$
- Question
  - Is InsertionSort $O(n^3)$?
  - Is InsertionSort O(n)?

Big O Fact

- A polynomial of degree k is $O(n^k)$
- Proof:
  - Suppose $f(n) = b_k n^k + b_{k-1} n^{k-1} + \ldots + b_1 n + b_0$
    - Let $a_i = | b_i |$
  - $f(n) \le a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$

$$\le \quad n^k \sum a_i \frac{n^i}{n^k} \quad \le \quad n^k \sum a_i \quad \le \quad cn^k$$

Lower Bound Notation

- We say InsertionSort's run time is $\Omega(n)$
- In general a function
  - $f(n)$ is $\Omega(g(n))$ if $\exists$ positive constants c and n_0 such that $0 \le c \cdot g(n) \le f(n) \quad \forall\ n \ge n_0$
- Proof:
  - Suppose run time is $an + b$
    - Assume a and b are positive (what if b is negative?)
  - $an \le an + b$

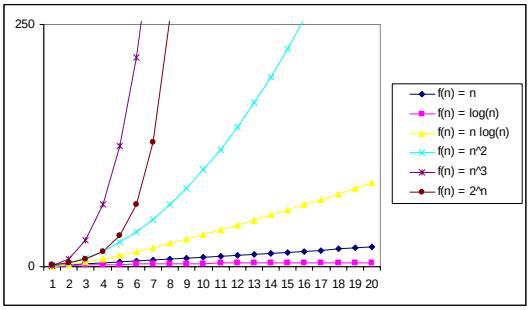
Asymptotic Tight Bound

- A function $f(n)$ is $\Theta(g(n))$ if $\exists$ positive constants c_1, c_2, and n_0 such that

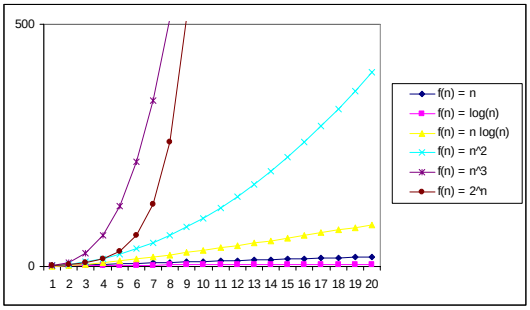
$$c_1\, g(n) \le f(n) \le c_2\, g(n) \quad \forall\ n \ge n_0$$

- Theorem
  - $f(n)$ is $\Theta(g(n))$ iff $f(n)$ is both $O(g(n))$ and $\Omega(g(n))$

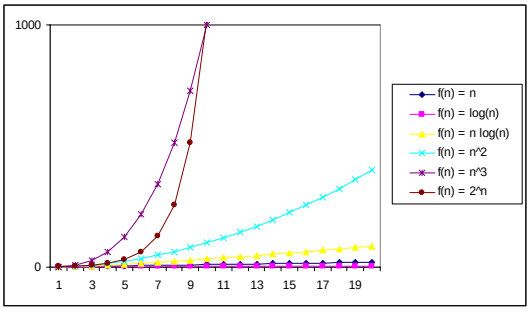
Practical Complexity



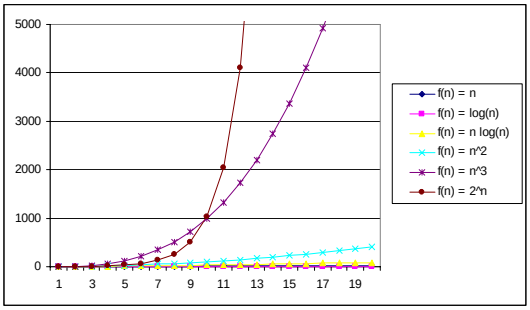
Practical Complexity



Practical Complexity



Practical Complexity



Other Asymptotic Notations

- A function f(n) is o(g(n)) if for any positive constant *c*, there exists n_0 such that
  $f(n) < c\ g(n)\ \forall\ n \geq n_0$

or $\lim_{n\to\infty} (f(n)/g(n)) = 0$

Other Asymptotic Notations

- A function f(n) is ω(g(n)) if for any positive constant *c*, there exists n_0 such that
  $c\ g(n) < f(n)\ \forall\ n \geq n_0$

or $\lim_{n\to\infty} (f(n)/g(n)) = \infty$

Comparison of functions

- Intuitively,
  - o() is like <
  - ω() is like >
  - Θ() is like =
  - O() is like ≤
  - Ω() is like ≥

Typical Running Time Functions

- Θ (1) (constant running time):
  - Instructions are executed once or a few times
- Θ (logN) (logarithmic)
  - A big problem is solved by cutting the original problem in smaller sizes, by a constant fraction at each step
- Θ (N) (linear)
  - A small amount of processing is done on each input element
- Θ (N logN)

Typical Running Time Functions

- Θ (N^2) (quadratic)
  - Typical for algorithms that process all pairs of data items (double nested loops)
- Θ (N^3) (cubic)
  - Processing of triples of data (triple nested loops)
- Θ (N^K) (polynomial)
- Θ (2^N) (exponential)
  - Few exponential algorithms are appropriate for practical