Automata (Spring 2009) Qual Exam

Answer ALL questions

Question 1 (10 points + 10 points)

(a) Let $p : U \longrightarrow \{T, F\}$ and $S \subseteq U$. Consider the following two statements:

$$\forall a \in S\ p(a)$$

and

$$\forall a\ ((a \in S) \wedge p(a)).$$

Note that $\forall a \in S\ p(a)$ states that $p(a)$ is true for all a in the set S.

Are the two statements equivalent? Justify your answer.

(b) Let $p : U \longrightarrow \{T, F\}$ and $S \subseteq U$. Consider the following two statements:

$$\neg(\exists a \in S\ p(a))$$

and

$$\forall a\ ((a \in S) \wedge (\neg p(a))).$$

Note that $\exists a \in S\ p(a)$ states that $p(a)$ is true for some a in the set S.

Are the two statements equivalent? Justify your answer.

Question 2 (15 points + 15 points)

(a) Give a context-free language L such that $\overline{L}$ is also context-free, but L is not regular. Justify your answer.

(b) Give a context-free language L such that $\overline{L}$ is not context-free. Justify your answer.

Question 3 (5 points + 15 points + 10 points + 20 points)

Let $L_n = \{xy \mid x, y \in \{0,1\}^n,\ x = y\}$. That is, L_n consists of strings that are of lengths $2n$ satisfying the condition that $w \in L_n$ iff the first and second halves of w are the same. Examples: $011011 \in L_3$, $100100 \in L_3$, $101011 \notin L_3$, $100110 \notin L_3$, $10111011 \in L_4$, $01000100 \in L_4$, $10110011 \notin L_4$.

(a) Argue that L_n is regular.

(b) Let L be a regular language and m be a positive integer. Let u_i and v_i, $i = 1, 2, \ldots, m$, be two sequences of m strings. Suppose $u_i v_j \in L$ if and only if $i = j$. Show that any NFA for L has at least m states. *Hint: Given an NFA for L, consider the accepting paths for $u_i v_i \in L$, $i = 1, 2, \ldots, m$.*

(c) Show that any NFA for L_n has 2^n states. Hint: use part (b).

(d) Give an $O(n^2)$-state NFA for $\overline{L_n}$.

ANSWERS (Sketch)

1 (a) Not equivalent. Counter example: Let $U = \{a, b\}$, $S = \{a\}$, $p(a) = T$ and $p(b) = F$.

1 (b) Not equivalent. Counter example: Let $U = \{a, b\}$, $S = \{a\}$, $p(a) = F$ and $p(b) = T$.

2 (a) $L = \{w \in \{0,1\}^* \mid w \text{ has the same number of 0's and 1's }\}$

2 (b) $\overline{L} = \{a^n b^n c^n \mid n \geq 0\}$

3 (a) L_n is a finite language (consisting of only a finite number of strings).

3 (b) Let M be an NFA for L. Consider an accepting path for $u_i v_i \in L$. Let q_i be the state reached at the middle of the path after the processing of u_i. One can see that all the q_i's are distinct for $1 \leq i \leq m$. If $q_i = q_j$, where $i \neq j$, the NFA would have accepted the string $u_i v_j \notin L$, a contradiction.

3 (c) For $0 \leq i \leq 2^n - 1$, define $u_i = v_i$ to be the n-bit binary string of value i.

3 (d) For $1 \leq i \leq n$, define $L'_i = \{xy \mid x, y \in \{0,1\}^n,\ x \text{ and } y \text{ differs in the } i\text{-th position}\}$. It is easy to construct an $O(n)$-state DFA M_i for L'_i. Also, let D_0 be the DFA that accepts all strings of length $\neq 2n$. We can combine nondeterministically the $n+1$ DFA's to denote $\overline{L_n} = \{0,1\}^{<2n} \cup \{0,1\}^{>2n} \cup L'_1 \cup L'_2 \ldots \cup L'_n$.