Ph.D. Qualifying Exam (Spring 2002)
Automata and Formal Languages
Answer ALL questions

Question 1.
Let $L = \{ww^{\text{rev}} \mid w \in \{a,b\}^*\}$. Note that w^{rev} denotes the reversal of w.

(a) (20%) Prove that L is not regular.

(b) (20%) Is L context-free? Give a context-free grammar for L if you answer 'yes'. Otherwise, prove that it is not context-free.

(c) (20%) Is $\bar{L} = \{a,b\}^* - L$ context-free? Give a context-free grammar for $\bar{L}$ if you answer 'yes'. Otherwise, prove that it is not context-free.

Question 2. (20%)
Let $L = \{ww \mid w \in \{a,b\}^*\}$. Is L context-free? Give a context-free grammar or a PDA for L if you answer 'yes'. Otherwise, prove that it is not context-free.

Question 3. (20%)
Let $C_n = \{x \mid x \text{ is a binary number that is a multiple of } n\}$. Note that x is given with the lowest-ordered digit first. For example, consider the decimal number 6. The binary number equivalent is 110. We assume that the binary number 110 is presented as 0 followed by 1 and 1. Show that C_n is regular.

Question 1.

(a) Suppose L is regular. Then there exists a DFA M to recognize L. Let $w_k = a^k b$ for $k \geq 0$. For $j > i \geq 0$, we claim that the two states reached by M on processing w_i and w_j should differ. If they are the same state, then M would either accept both or reject both $w_i ba^i$ and $w_j ba^i$. But this contradicts the assumption that M recognizes L since $w_i ba^i \in L$ and $w_j ba^i \notin L$. Thus, M goes to different states for different w_k's where $k \geq 0$. Hence M has infinite states, a contradiction.

(b) The context-free grammar is $S \rightarrow \epsilon \mid aSa \mid bSb$.

(c) The context-free grammar is $S \rightarrow aSa \mid bSb \mid a \mid b \mid aAb \mid bAa$, $A \rightarrow \epsilon \mid aA \mid bA$.

Question 2.

L is not context-free. The proof by using the pumping lemma is given in Example 2.22 (page 119) of the textbook.

Question 3.

The state set $Q = \{0, 1, 2, ..., n-1\} \times \{0, 1, 2, ..., n-1\}$. The transition function $\delta : Q \times \{0, 1\} \rightarrow Q$ is defined as $\delta((v, w), x) = ((v + x * w)\%n, (2 * w)\%n)$. The starting state is $(0, 1)$ and the set of final states is $\{(0, 0), (0, 1), (0, 2), ..., (0, n-1)\}$.