Department of Computer Science Fall 2006 New Mexico State University

Ph.D. Qualification Exam: Analysis of Algorithms

This is a closed book exam. The total of points is 100. Please answer all questions.

(20 points) 1. Determine asymptotically tight lower and upper bounds for

$$T(n) = T\left(\left\lceil \frac{1}{5}n \right\rceil\right) + T\left(\left\lfloor \frac{4}{5}n \right\rfloor\right) + n, \qquad n > 1,$$

with $T(1) = b$.

2. Let $l_1, l_2, \ldots, l_N$ be a sequence of N labels, each taking a value of either A or B. One inserts at most $M(< N)$ boundaries in between the labels to form $M+1$ intervals. The $M+1$ intervals are numbered by $1, 2, \ldots, M+1$ with smaller values representing intervals of smaller index labels. The goal is to separate distinct labels into different intervals as much as possible. You CANNOT change the ordering of the labels. One way to achieve the goal is to define

$$f(M \text{ boundaries}) = \sum_{m=1}^{M+1} \max\{\#A_m, \#B_m\}$$

where $\#A_m$ and $\#B_m$ are the total numbers of A and B labels in interval m, respectively. Then find M boundaries to maximize $f(M \text{ boundaries})$, i.e.,

$$\max_{M \text{ boundaries}} f(M \text{ boundaries})$$

Here is an example of 6 labels to be grouped into 3 intervals ($M = 2$).

$$\begin{array}{cccccc} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ A & A & B & B & A & A \end{array}$$

The first grouping

$$\begin{array}{c|cc|ccc} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ A & A & B & B & A & A \end{array}$$

is not optimal with $f(2 \text{ boundaries}) = 4$.

The second grouping

$$\begin{array}{cc|cc|cc} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ A & A & B & B & A & A \end{array}$$

is optimal with $f(2 \text{ boundaries}) = 6$.

(15 points) (a) Will the following greedy approach guarantee an optimal solution?

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GREEDY-DISCRETIZATION(l, N, M)
n ← 1
m ← 1
while m ≤ M and n < N
    do if l_{n+1} ≠ l_n
         then create the m-th boundary between l_n and l_{n+1}
              m ← m + 1
       n ← n + 1
return all m − 1 boundaries
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(25 points) (b) Design an algorithm using dynamic programming to produce optimal intervals. Please provide the recurrence equation and give the running time.

Hint: Optimal m intervals will require $m-1$ previous intervals to be optimal. You may want to compute the following quantity by dynamic programming

$$W[i,m] = \max_{m-1 \text{ boundaries on } l_1, l_2, \ldots, l_i} \sum_{q=1}^{m} \max\{\#A_q, \#B_q\}.$$

3. This problem addresses the number of bitwise multiplications needed when two n-bits binary integers x and y are multiplied.

(10 points) (a) How many bit-wise multiplications are needed in the brute-force way of doing $x \cdot y$?

(20 points) (b) Please design a divide-and-conquer solution to compute $x \cdot y$ with asymptotically fewer bitwise multiplications. Justify why your strategy saves on bitwise multiplications.

Hint: You may take advantage of the following equation during your derivation:

$$(x_1 + x_2) \cdot (y_1 + y_2) = x_1 \cdot y_1 + x_1 \cdot y_2 + x_2 \cdot y_1 + x_2 \cdot y_2$$

(10 points) (c) If one also takes bitwise additions into consideration, how does that affect the overall asymptotic running time for the divide-and-conquer approach in part (b)?