

Ph.D. Qualifying Exam (Fall 2004)  
Automata and Formal Languages  
Answer ALL questions

Consider the following pseudo code for processing inputs of strings over  $\{0,1\}$ .

```
int state = 0; // state is initially set to 0
                // there are 3 possible states: 0,1,2

IntStack S;    // S is a integer stack of 0's and 1's

int i;         // an integer variable holding either 0 or 1

main() {
  initStack(S);
  while ( not end-of-input() ) {

    push( S, getInput() );
    // getInput returns the next input symbol
    // which is either the integer 0 or 1
    //
    // Push the input symbol into the stack S.

  }

  while ( not IsStackEmpty(S) ) {
    i = Pop(S); // the top item in S is popped and saved in i

    if (state == 0 and i == 0)
      state = 2;
    else
      state = (state + i) mod 3;
  }

  if (state == 2)
    return 1; // input is accepted
  else
    return 0; // input is rejected
}
```

Let  $L_1 \subseteq \{0,1\}^*$  be the language recognized by the pseudo-code program.

Let  $L_2 = \{x0^i y \mid i > |x| \text{ or } i > |y| \text{ where } x, y \in \{0, 1\}^*\}$ .

Let  $L_3 = \{x0^i y \mid i > |x| \text{ and } i > |y| \text{ where } x, y \in \{0, 1\}^*\}$ .

Question 1. (30 points)

Is  $L_1$  regular?

If your answer is yes,  
you are required to give a DFA (**deterministic** finite automaton) for  $L_1$ .

If your answer is no, you should provide a proof that  $L_1$  is not regular.

Question 2. (20 points)

Is  $L_2$  regular?

If your answer is yes,  
you are required to give a DFA (**deterministic** finite automaton) for  $L_2$ .

If your answer is no, you should provide a proof that  $L_2$  is not regular.

Question 3. (20 points)

Is  $L_2$  context-free?

If your answer is yes, you are required to give a CFG (context-free grammar) for  $L_2$ .

If your answer is no, you should provide a proof that  $L_2$  is not context-free.

Question 4. (30 points)

Is  $L_3$  context-free?

If your answer is yes, you are required to give a CFG (context-free grammar) for  $L_3$ .

If your answer is no, you should provide a proof that  $L_3$  is not context-free.

Answers:

Question 1.

$L_1$  is regular. The language  $L_1$  recognized by the program is the reversal of the language of the following DFA of three states 0, 1 and 2, where 0 is the start state and 2 is the accepting state, and the transition function is given by  $\delta(0, 0) = 2, \delta(0, 1) = 1, \delta(1, 0) = 1, \delta(1, 1) = 2, \delta(2, 0) = 2, \delta(2, 1) = 0$ . We can reverse the transitions of the DFA to obtain the NFA with 2 as the start state, 0 as the accepting state, and the transition function as given by  $\delta(0, 0) = \{\}, \delta(0, 1) = \{2\}, \delta(1, 0) = \{1\}, \delta(1, 1) = \{0\}, \delta(2, 0) = \{0, 2\}, \delta(2, 1) = \{1\}$ . Determinizing the NFA using the subset construction gives a DFA with 8 subset states:  $\delta(\{\}, 0) = \{\}, \delta(\{\}, 1) = \{\}, \delta(\{0\}, 0) = \{\}, \delta(\{0\}, 1) = \{2\}, \delta(\{1\}, 0) = \{1\}, \delta(\{1\}, 1) = \{0\}, \delta(\{2\}, 0) = \{0, 2\}, \delta(\{2\}, 1) = \{1\}, \delta(\{0, 1\}, 0) = \{1\}, \delta(\{0, 1\}, 1) = \{0, 2\}, \delta(\{0, 2\}, 0) = \{0, 2\}, \delta(\{0, 2\}, 1) = \{1, 2\}, \delta(\{1, 2\}, 0) = \{0, 1, 2\}, \delta(\{1, 2\}, 1) = \{0, 1\}, \delta(\{0, 1, 2\}, 0) = \{0, 1, 2\}, \delta(\{0, 1, 2\}, 1) = \{0, 1, 2\}$  where  $\{2\}$  is the start state, and  $\{0\}, \{0, 1\}, \{0, 2\}, \{0, 1, 2\}$  are accepting states.

Question 2.

$L_2$  is not regular. Suppose the contrary that there is DFA  $M_2$  for  $L_2$ . Consider the strings  $w_i = 0^i$  for  $i = 1, 2, \dots$ . We claim that  $M_2$  on processing  $w_i$  will arrive at different states for different  $i$ 's. If  $M_2$  arrives at the same state for processing  $w_i$  and  $w_j$  where  $i < j$ , then  $M_2$  will either accept both  $w_i 1^{j-1}$  and  $w_j 1^{j-1}$  or reject both  $w_i 1^{j-1}$  and  $w_j 1^{j-1}$ . But, it is known that  $w_i 1^{j-1} \notin L_2$  and  $w_j 1^{j-1} \in L_2$ . Thus, the claim is proved and  $M_2$  has an infinite number of states, which contradicts the assumption that  $M_2$  is a finite automaton.

Question 3.

$L_2$  is context-free. We give a CFG for  $L_2$  as follows:  $S \rightarrow XD \mid DY, D \rightarrow \epsilon \mid 0D \mid 1D, X \rightarrow X0 \mid X'0, X' \rightarrow \epsilon \mid 0X'0 \mid 1X'0, Y \rightarrow 0Y \mid 0Y', Y' \rightarrow \epsilon \mid 0Y'0 \mid 0Y'1$ , where  $S$  is the start symbol.

Question 4.

$L_3$  is not context-free. We prove that the statement of the pumping lemma is not observed for  $L_3$ . Let  $p$  be the pumping constant. Pick the string  $1^p 0^{p+1} 1^p \in L_3$ . We call the first  $p$  1's as region 1, the next  $p + 1$  0's as region 2, and the last  $p$  1's as region 3. Let  $uvwxy = 1^p 0^{p+1} 1^p$ . Case 1:  $vw$  lies in region 1. Action: pumping up. Case 2:  $vw$  lies in region 2. Action: pumping down. Case 3:  $vw$  lies in region 3. Action: pumping up. Case 4:  $vw$  lies in both regions 1 and 2. Action: pumping down. Case 5:  $vw$  lies in both regions 2 and 3. Action: pumping down. In all the five cases, the resulting string after pumping is not in  $L_3$ .