Ph.D. Qualifying Exam (Fall 2001)
Automata and Formal Languages
Answer ALL questions

Question 1.
Consider the language $L = \{0^i 1^j \mid i \geq j \geq 1\}$.

**(a)** (20%) Is $L$ regular? Provide a proof for your answer.

**(b)** (15%) Give a context-free grammar for $L$.

**(c)** (15%) Give a *deterministic* pushdown automaton for $L$.

**(d)** (25%) Mechanically, convert the PDA that you obtained in (c) into an equivalent context-free grammar. Show the steps.

Question 2. (25%)
Consider the language $L = \{0^i 1^j 0^i \mid i \geq j \geq 1\}$. Show that $L$ is not context-free.

Solutions

Q1(a). Suppose $L$ is regular and let $M$ be a DFA for $L$. Claim: $M$ goes to different states for processing $0^i$ for different $i \geq 1$. Thus, $M$ has infinite number of states, a contradiction. Proof of claim: Assume the contrary that there exist $1 \leq i < j$ such that $\delta(q_0, 0^i) = \delta(q_0, 0^j)$. Next, $\delta(q_0, 0^i 1^j) = \delta(\delta(q_0, 0^i), 1^j) = \delta(\delta(q_0, 0^j), 1^j) = \delta(q_0, 0^j 1^j) \in F$. We conclude that $0^i 1^j \in L$, a contradiction because $i < j$.

Q1(b). $S \to AB$, $A \to \epsilon \mid 0A$, $B \to 01 \mid 0B1$

Q1(c). Both $q_0$ and $q_1$ are accepting states.
<consume 0, state $q_0$, top stack symbol \$, push $X$ onto stack, goto state $q_0$ >
<consume 0, state $q_0$, top stack symbol $X$, push $X$ onto stack, goto state $q_0$ >
<consume 1, state $q_0$, pop top stack symbol $X$, goto state $q_1$ >
<consume 1, state $q_1$, pop top stack symbol $X$, goto state $q_1$ >

Q1(d). We can turn the DPDA into a PDA that accepts input by empty stack. This is done by adding the following two instructions:

<state $q_1$, pop top stack symbol $X$, goto state $q_1$ >
<state $q_1$, pop top stack symbol \$, goto state $q_1$ >

Note that the PDA is a nondeterministic machine. Next, convert the PDA into an equivalent one-state PDA which also accepts inputs by empty stack.

<consume 0, pop $[q_0, \$]$, push $[q, \$]$, push $[q_0, X, q]$ > where $q = q_0, q_1$.
<consume 0, pop $[q_0, X, q']$, push $[q, X, q']$, push $[q_0, X, q]$ > where $q = q_0, q_1$ and $q' = q_0, q_1$
<consume 1, pop $[q_0, X, q_1]$ >
<consume 1, pop $[q_1, X, q_1]$ >
<pop $[q_1, X, q_1]$ >
<pop $[q_1, \$]$ >

The corresponding CFG is:

$$
\begin{aligned}
&[q_0, \$] &&\longrightarrow 0[q_0, X, q][q, \$] \text{ where } q = q_0, q_1 \\
&[q_0, X, q'] &&\longrightarrow 0[q_0, X, q][q, X, q'] \text{ where } q = q_0, q_1 \text{ and } q' = q_0, q_1 \\
&[q_0, X, q_1] &&\longrightarrow 1 \\
&[q_1, X, q_1] &&\longrightarrow 1 \\
&[q_1, X, q_1] &&\longrightarrow \epsilon \\
&[q_1, \$] &&\longrightarrow \epsilon
\end{aligned}
$$

Q2. (Sketch) Similar to the standard example for showing that $\{a^n b^n c^n \mid n \geq 0\}$ is not context-free. Pick $s = 0^p 1^p 0^p$ where $p$ is the pumping constant. Always "Pump up" instead of "pump down".