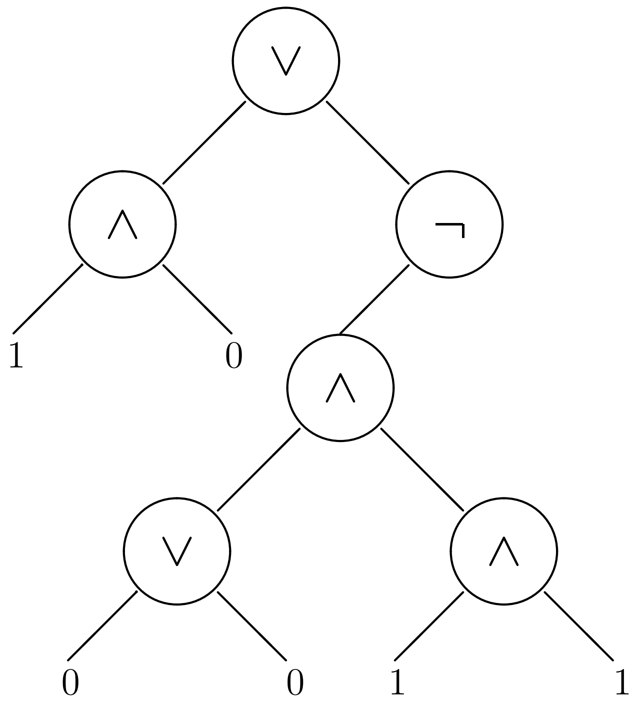
Ph.D. Qualifying Exam (Fall 2000)
Automata and Formal Languages
Answer ALL questions

Question 1.
We consider the expression tree for a boolean expression with operators $\vee$ (or), $\wedge$ (and) and $\neg$ (not). An example is given below:



An evaluation of the expression gives the value 1. A preorder traversal of the tree returns the string $\vee \wedge 1\ 0 \neg \wedge \vee\ 0\ 0 \wedge 1\ 1$. Let L be the set of strings returned by the preorder traversals of all (boolean) expression trees that are evaluated to 1.

(a) (25%) Show that L is not regular.

(b) (25%) Give a context-free grammar for L.

(c) (25%) Give a *deterministic* pushdown automaton for L. Besides giving a formal presentation of a deterministic PDA, you are encouraged (though not required) to provide an informal discussion of the design of the automaton.

Question 2. (25%)
Let $\Sigma = \{0, 1, +, =\}$ and
$ADD = \{x = y + z \mid x, y, z$ are binary integers, and x is the sum of y and $z\}$.
For example, $10100 = 1001 + 1011 \in ADD$ and $110 = 10 + 1 \notin ADD$.
Show that ADD is not context-free. (Hint: consider strings $x = 0 + x \in ADD$.)

Answers:

1. (a)
Suppose the contrary that L is regular and there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ for L. Consider the strings $w_k = \wedge^k$, for $k \geq 1$. We claim for $1 \leq i < j$, $\delta(q_0, w_i) \neq \delta(q_0, w_j)$. If $\delta(q_0, w_i) = \delta(q_0, w_j)$, then $\delta(q_0, w_j 1^{i+1}) = \delta(\delta(q_0, w_j), 1^{i+1})$ $= \delta(\delta(q_0, w_i), 1^{i+1}) = \delta(q_0, w_i 1^{i+1}) = \delta(q_0, \wedge^i 1^{i+1}) \in F$ which contradicts with the fact that $\wedge^j 1^{i+1} \notin L$ when $i \neq j$. Thus, M has infinite states, a contradiction.

1 (b)
$S_1 \longrightarrow \wedge S_1 S_1 \mid \vee S_0 S_1 \mid \vee S_1 S_0 \mid \vee S_1 S_1 \mid \neg S_0 \mid 1$
$S_0 \longrightarrow \vee S_0 S_0 \mid \wedge S_0 S_0 \mid \wedge S_0 S_1 \mid \wedge S_1 S_0 \mid \neg S_1 \mid 0$

1. (c)
Let q be the starting state and q_a be the accepting state. The transitions are:
if (state q, empty stack, input 1) then (state = q_1)
if (state q, input x is an operator) then (push x)
if (state q, stack top = operator, input 0) then (state = q_0)
if (state q, stack top = operator, input 1) then (state = q_1)
if (state q_0, stack top = $\neg$) then (state = q_1, pop stack)
if (state q_1, stack top = $\neg$) then (state = q_0, pop stack)
if (state q_0, stack top = 0) then (state = $q_{0,0}$, pop stack)
if (state q_0, stack top = 1) then (state = $q_{0,1}$, pop stack)
if (state q_1, stack top = 0) then (state = $q_{0,1}$, pop stack)
if (state q_1, stack top = 1) then (state = $q_{1,1}$, pop stack)
if (state q_0, stack top = operator, input x is an operator) then (state = q, push 0, push x)
if (state q_1, stack top = operator, input x is an operator) then (state = q, push 1, push x)
if (state $q_{0,0}$, top stack = $\wedge$ or $\vee$) then (state = q_0, pop stack)
if (state $q_{0,1}$, top stack = $\wedge$) then (state = q_0, pop stack)
if (state $q_{0,1}$, top stack = $\vee$) then (state = q_1, pop stack)
if (state $q_{1,1}$, top stack = $\wedge$ or $\vee$) then (state = q_1, pop stack)
if (state q_1, stack empty, end-of-input) then (state = q_a)

2. (The solution is similar to that of Example 2.22 in Sipser's book.) (Sketch) Choose s to be $1^p 0^p = 0 + 1^p 0^p$ where p is the pumping constant. Note that we assume that x, y and z cannot be empty strings. That is, empty string is not a binary integer. When pumping, always pump down by taking $i = 0$. One case of the pumping is when $u = 0^p 1^p =$, $v = 0$, $x = \epsilon$, $y = \epsilon$ and $z = +1^p 0^p$. Another case of the pumping is when $u = 0^p 1^p =$, $v = \epsilon$, $x = \epsilon$, $y = 0$ and $z = +1^p 0^p$. Of course, there are still many other cases.