

Peano Arithmetic

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In this project we will learn the first-order theory of arithmetic, known as *Peano Arithmetic*. The formal development of arithmetic goes all the way back to ancient Greek mathematics. But the modern theory of arithmetic was developed only in the second half of the nineteenth century with the work of Hermann Grassmann (1809–1877), Richard Dedekind (1833–1916), and Gottlob Frege (1848–1925). However, it was not until Giuseppe Peano’s (1858–1932) treatise “*Arithmetices principia, nova methodo exposita*” (The principles of arithmetic, presented by a new method) that the axiomatic theory of arithmetic was devised as we know it today.¹

Giuseppe Peano was born on 27 August of 1858 near Turin, in the village of Spinetta. He went to village school first in Spinetta and then in nearby Cuneo. In the early seventies he continued his studies in Turin. In 1876 Giuseppe enrolled in the University of Turin, from which he graduated in 1880 with ‘high honors’.

Peano spent his entire academic career at the University of Turin, starting as an assistant in 1880, and steadily progressing through the ranks all the way until he reached the rank of a full chair, which he held from 1895 until his death in 1932. In 1886–1901 he also held a position of professor at the Royal Military Academy of Turin.

Peano’s main contributions were in analysis, axiomatization of mathematics, and most importantly in mathematical logic. In fact, in the late nineteenth and early twentieth centuries he was considered the leading figure in mathematical logic alongside Frege and Bertrand Russell (1872–1970).

In analysis Peano is best known for his space-filling curve, which bears his name. But he also contributed to differential equations and measure theory. In fact, Henri Lebesgue (1875–1941) acknowledged Peano’s influence on his own groundbreaking research. Peano was a great proponent of Grassmann’s revolutionary development of linear algebra. Actually, Peano was one of the first who realized the importance of Grassmann’s work. His book [6] gives the first axiomatic development of vector spaces. It is remarkable and not well known that Peano was the inventor of the symbol “ \in ” that we commonly use today to denote the set membership relation. But perhaps the most widely known of Peano’s accomplishments is his set of postulates for the natural numbers, which is the subject of this project.

Peano was a very prolific author. Throughout his professional career he published over 200 research papers and books. He was very actively involved in editorial work as well. In 1891 he founded the journal *Rivista di Matematica*, and was the main force behind cataloguing all the known theorems on the then mainstream subjects of mathematics—the project that became known as the *Formulario* project.

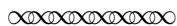
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¹It has to be noted that Grassmann’s treatise [4] was the earliest, and appeared over 20 years before the work of Frege, Dedekind, and Peano. On the other hand, the treatises of Frege [3], Dedekind [2], and Peano [7] appeared within five years of each other, with Frege’s publication in 1884 being the earliest, and Peano’s in 1889 the latest. Although the work of Frege, Dedekind, and Peano have a lot in common, it follows from their own testaments that they were not aware of each others work at the time of completion of their own.

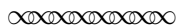
Peano created his own school of mathematics, the best known representatives of which are Cesare Burali-Forti (1861–1931), Mario Pieri (1860–1913), and Alessandro Padoa (1868–1937). He also actively corresponded with the leading mathematicians of his time, including Georg Cantor (1845–1918), Frege, Felix Klein (1849–1925), Russell and many others. His controversy with another famous professor at Turin, Vitto Volterra (1860–1940), is well-documented in [5, Chapter 8].

During his life Peano was well recognized as a mathematician in Italy. In 1891 he was elected a member of the Academy of Sciences of Turin, and in 1905 he became a Corresponding Member of the *Accademia dei Lincei*—considered as one of the highest honors for an Italian scientist. In addition, he was first made a Knight of the Crown of Italy (1895), then an Officer of the Crown of Italy (1917), and finally *Commendatore* of the Crown of Italy (1921).

The international influence of Peano was perhaps best manifested at the 1900 International Congress of Philosophy in Paris, where Peano and his school dominated the discussion. The following is Russell’s account of Peano at the congress and his effect on Russell, taken from [8, pp. 217–218]:



The Congress was a turning point of my intellectual life, because I there met Peano... In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked.



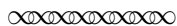
Although the congress was the peak of Peano’s prowess, it was also the starting point of his decline as a mathematician. His interests started to switch slowly towards artificial languages and later also towards mathematics education. At the time artificial languages were in fashion. Among the most popular ones were *Esperanto* and *Volapük*. Instead Peano proposed to use a simplified form of Latin, Latin stripped of grammar—*Latino sine flexione*. In 1908 he was unanimously elected director of the *Academia pro Interlingua*—an international academy of artificial languages—and remained its director until his death in 1932. Since his election his interests were mostly directed towards promoting the international auxiliary language movement and away from original mathematical research.

Peano died on 20 April, 1932. His last years were serene and satisfying. He was a well respected and revered figure in the mathematical community and international auxiliary language movement. Peano’s gentle personality, his tolerance of human weakness, and his perennial optimism have been remembered by his family, friends, and disciples. For the rest of us, however, his name will always be associated with the set of postulates for the natural numbers, which we discuss now in detail.

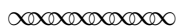
In 1889 Peano published a treatise “*Arithmetices principia, nova methodo exposita*” (The principles of arithmetic, presented by a new method), where his famous postulates for the natural numbers appeared for the first time. This project is based on its English translation which appeared in [1, pages 101–134].

The treatise consists of a preface and 10 sections. In the preface Peano explains his formalism and discusses the basic principles of logic and set theory that he uses throughout the treatise. Sections 1 through 7 develop the axiomatic theory of natural numbers and are our main concern. Sections 8 and 9 are dedicated to the theory of rational and real numbers, and finally section 10 presents some new theorems about sets of real numbers.

We begin by reading several passages from the preface to become familiar with Peano’s notation. From the outset Peano states that his development is purely formal:



I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs. The signs pertain either to logic or to arithmetic.



A little later he clearly expresses his belief in the axiomatic method:

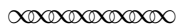


But in order to treat other theories, it is necessary to adopt new signs to indicate new entities. I believe, however, that with only these signs of logic the propositions of any science can be expressed, so long as the signs which represent the entities of the science are added.

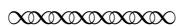


A belief that the twentieth century proved to be true!

In his logical notation Peano uses “P” for *proposition*, “ \cap ” for *and*, “ \cup ” for *or*, “ $-$ ” for *not*, “ Λ ” for *false* or *absurd*, and “ \supset ” for *one deduces*. For the sake of precision, Peano sometime uses “ $\supset_{x,y,\dots}$ ” instead of “ \supset ”:



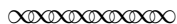
If the propositions a, b contain the indeterminate quantities x, y, \dots , that is, express conditions on these objects, then $a \supset_{x,y,\dots} b$ means: whatever the x, y, \dots , from proposition a one deduces b . If indeed there is no danger of ambiguity, instead of $a \supset_{x,y,\dots} b$ we write only \supset .



In his set theoretic notation Peano uses “K” for *class* (or *set* as we say today), and for the first time(!) introduces “ \in ” to denote *is a member of*. In addition, he uses the same symbols “ \supset ” and “ Λ ” that he used in his logical notation to denote *is a subset of* and *empty set*, respectively.² He also uses $[x \in]a$ for *those x such that a*. In his formalism Peano frequently uses dots instead of parentheses:



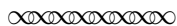
Generally we write signs on the same line. So that it will be clear how they are to be joined, we use parentheses, as in algebra, or rather points $. : . : ::$ etc. So that a formula divided by points may be understood, first the signs which are not separated by points are taken together, then those separated by one point, then those by two points, etc. For example, let a, b, c, \dots be any signs. Then $ab . cd$ means $(ab)(cd)$; and $ab . cd : ef . gh : . : k$ means $((ab)(cd))((ef)(gh))k$.



After explaining his notation in the preface, Peano begins section 1 by presenting his famous postulates for the natural numbers.³

²We note that the modern day notation for a subset relation is \subset , which is the opposite of Peano’s notation. Also, today we denote the empty set by \emptyset .

³In fact, Peano had nine axioms, but four of the nine were expressing the usual properties of $=$ (such as $=$ is reflexive, symmetric, and transitive), which we decided to omit from the list below.



Explanations

The sign \mathbb{N} means *number (positive integer)*; 1 means *unity*; $a + 1$ means the *successor of a or a plus 1*; and $=$ means *is equal to...*

Axioms

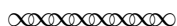
$$1 \in \mathbb{N}.$$

$$a \in \mathbb{N} \supset a + 1 \in \mathbb{N}.$$

$$a, b \in \mathbb{N} \supset : a = b \Rightarrow a + 1 = b + 1.$$

$$a \in \mathbb{N} \supset a + 1 - 1 = a.$$

$$k \in \mathbb{K} \therefore 1 \in k \therefore x \in \mathbb{N} \cdot x \in k \supset x + 1 \in k \therefore \mathbb{N} \supset k.$$



Task 1. Describe each postulate in your own words.

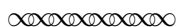
Note: Although Peano did not include 0 in \mathbb{N} , today it is customary to do so.

Task 2. Taking the above note into account, first describe the symbol set S , and then express the five Peano postulates in the first-order language \mathcal{L}^S . How many axioms do you need to express the Peano postulates in \mathcal{L}^S ? Explain why.

Task 3. Do the Peano postulates have a model? Justify your answer.

Task 4. Are the Peano postulates independent? Justify your answer.

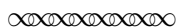
Peano's next task is to give a recursive definition of addition:



Definition

$$a, b \in \mathbb{N} \supset a + (b + 1) = (a + b) + 1.$$

NOTE This definition should be read: if a and b are numbers, and $(a + b) + 1$ has meaning (that is, if $a + b$ is a number), but $a + (b + 1)$ has not yet been defined, then $a + (b + 1)$ indicates the number that follows $a + b$.



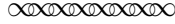
Task 5. Expand the symbol set S by a new binary function symbol $+$ for addition; that is, set $T = S \cup \{+\}$. In the enriched first-order language \mathcal{L}^T express the axioms defining addition.

At the end of section 1, Peano proves that $+$ is well defined (that is, $a, b \in \mathbb{N}$ imply $a + b \in \mathbb{N}$), that $+$ is associative (that is, $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{N}$), and that $+$ is commutative (that is, $a + b = b + a$ for all $a, b \in \mathbb{N}$).

Task 6. Let Φ denote the set of Peano postulates in \mathcal{L}^S together with the two axioms in \mathcal{L}^T defining $+$. Show that it is derivable from Φ that $+$ is associative.

Task 7. Show that it is derivable from Φ that $+$ is commutative. (Hint: First show that $(\forall x)(x + 0 = 0 + x)$ and $(\forall x)(x + 1 = 1 + x)$ are derivable from Φ , where 1 is an abbreviation of $s(0)$. To show that $(\forall x)(x + 1 = 1 + x)$ is derivable from Φ , first show that $(\forall x)(s(x) = x + s(0))$ is derivable from Φ .)

In section 2 Peano defines subtraction and $<$.



Explanations

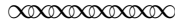
The sign $-$ is read *minus*, $<$ is read *is less than*, and $>$ is read *is greater than*.

Definitions

$$a, b \in \mathbb{N} . \supset : b - a = \mathbb{N}[x \in](x + a = b).$$

$$a, b \in \mathbb{N} . \supset : a < b . = . b - a - = \Lambda.$$

$$a, b \in \mathbb{N} . \supset : b > a . = . a < b.$$

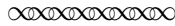


Task 8. Describe the above three definitions in your own words.

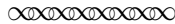
Note: Since subtraction is only a partial function, in \mathcal{L}^T it is easier to define $<$ directly from $+$.

Task 9. Define $<$ in \mathcal{L}^T .

In the remainder of section 2 Peano proves many basic properties of $<$. In particular, he shows:



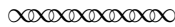
$$a, b, a', b' \in \mathbb{N} . a < b . a' < b' : \supset . a + a' < b + b'.$$



And proves that $<$ is a strict linear order.

Task 10. Express the above theorem in \mathcal{L}^T and show that it is derivable from Φ . Also show that it is derivable from Φ that $<$ is a strict linear order.

In a short section 3 Peano discusses maxima and minima of subsets of \mathbb{N} . We will not discuss section 3 here. Peano begins section 4 by giving a recursive definition of multiplication.



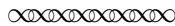
Definitions

$$a \in \mathbb{N} . \supset . a \times 1 = a.$$

$$a, b \in \mathbb{N} . \supset . a \times (b + 1) = a \times b + a.$$

$$ab = a \times b.$$

$$ab + c = (ab) + c.$$

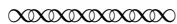


Task 11. Expand further the symbol set T by a new binary function symbol \times for multiplication; that is, set $U = T \cup \{\times\} = S \cup \{+, \times\}$. In the enriched first-order language \mathcal{L}^U express the axioms defining multiplication.

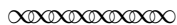
Let PA denote the set Φ together with the two axioms in \mathcal{L}^U defining multiplication. The first-order theory PA is called *Peano Arithmetic*.

Note: PA is strictly stronger than the first-order theory of arithmetic developed without addition and/or multiplication.

Peano shows that multiplication is well defined and establishes many basic properties of multiplication, such as commutativity, associativity, distributivity over $+$, and others. In particular, he proves the following theorems:

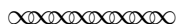


$a, b \in \mathbb{N} . \supset . ab = ba.$
 $a, b, c \in \mathbb{N} . \supset . a(b + c) = ab + ac.$
 $a, b, a', b' \in \mathbb{N} . a < a' . b < b' : \supset : ab < a'b'.$
 $a, b, c \in \mathbb{N} . \supset . a(bc) = (ab)c.$



Task 12. First describe the above four theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA. (Hint: In showing that it is provable in PA that multiplication is commutative, associative, and distributes over addition, start by showing that $\text{PA} \vdash (\forall x, y, z)(x(yz) = (xy)z)$ and $\text{PA} \vdash (\forall x, y, z)(x(y + z) = xy + xz)$. Then show that $\text{PA} \vdash (\forall x)(0x = 0)$ and $\text{PA} \vdash (\forall x)(x1 = 1x)$. Finally, prove that $\text{PA} \vdash (\forall x, y)(xy = yx)$.)

In a short section 5 Peano gives a recursive definition of powers and proves their basic properties. We will not discuss section 5 here. Peano begins section 6 by defining division.

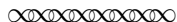


Explanations

The sign / is read *divided by*, D is read *divides*, or *is a divisor of*, \sqsupset is read *is a multiple of*, Np is read *prime number*, and π is read *is prime with*.

Definitions

$a, b \in \mathbb{N} . \supset . b/a = \mathbb{N}[x \in](xa = b).$
 $a, b \in \mathbb{N} . \supset : aDb . = . b/a - = \Lambda.$
 $a, b \in \mathbb{N} . \supset : b\sqsupset a . = . aDb.$
 $\text{Np} = \mathbb{N}[x \in](\exists D x . \exists > 1 . \exists < x : = \Lambda).$
 $a, b \in \mathbb{N} . \supset :: a\pi b . \therefore = \therefore \exists D a . \exists D b . \exists > 1 : = \Lambda.$

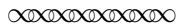


Task 13. Describe the above definitions in your own words.

Note: Since division is only a partial function, we will avoid its treatment in the first-order language of PA.

Task 14. Express D, \sqsupset , Np, and π in the first-order language of PA. (Hint: Note that Peano's definition of prime numbers does *not* rule out that 1 is a prime number. Nevertheless, in expressing Np, you should follow the standard assumption that 1 is *not* a prime number.)

Peano concludes section 6 by establishing many basic properties of division. In particular, he proves the following theorems:

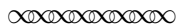


$a \in \mathbb{N} . \supset . 1Da.$
 $a \in \mathbb{N} . \supset . aDa.$
 $a, b, c \in \mathbb{N} . aDb . bDc : \supset . aDc.$
 $a, b, c, m, n \in \mathbb{N} . cDa . cDb : \supset . cDma + nb.$



Task 15. First describe the above theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA.

In a short section 7 Peano gives an extensive list of well-known theorems about integers that can be derived from his postulates, including several theorems from Euclid's *Elements* and Fermat's Little Theorem. The following theorems are part of the list:



- $x \in \mathbb{N} . \supset . x(x + 1) \text{D}2.$
- $x \in \mathbb{N} . \supset . x(x + 1)(x + 2) \text{D}6.$
- $a, b \in \mathbb{N} . a \in \mathbb{N}p . a - Db : \supset : a\pi b.$
- $a, b, c \in \mathbb{N} . a\pi b . cDa : \supset . c\pi b.$



Task 16. First describe the above theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA.

The last three sections of Peano's treatise deal with rational numbers, real numbers, and sets of real numbers. We will not discuss them here.

References

- [1] *Selected works of Giuseppe Peano*, University of Toronto Press, Toronto, Ont., 1973, Translated from the Italian and edited, with a biographical sketch and bibliography, by Hubert C. Kennedy.
- [2] Richard Dedekind, *Was sind und was sollen die Zahlen?*, Vieweg, Braunschweig, 1888.
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- [5] Hubert C. Kennedy, *Peano*, Studies in the History of Modern Science, vol. 4, D. Reidel Publishing Co., Dordrecht, 1980, Life and works of Giuseppe Peano.
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