An Introduction to Naive Set Theory and The Concept of Infinity: Guided by an Essay of Richard Dedekind

Joel G. Lucero-Bryan

May 11, 2009

An Introduction

During the last 100 to 150 years, it has become common practice for mathematicians to consider collecting numbers (and/or other objects) into a single conglomerate. This point of view in fact permeates much of higher level mathematics. As such, students of mathematics should have a firm grip on the underlying ideas of how these conglomerates behave. These conglomerates are typically called sets in modern mathematical language.

This project introduces the intuitive or naive point of view on sets. In fact our exploration will be guided by the famous mathematician Richard Dedekind (1831-1916). Before continuing, it is important to point out that a rigorous definition of a set, or as Dedekind says a system of elements, is usually avoided by textbooks. This is for a couple of reasons. First, many early attempts at defining what a set is or should be lead to antinomies or paradoxes; Dedekind's definition is not an exception. Despite this, Dedekind's description of the behavior of sets is eloquent, enlightening, and it exposes essential mechanics of sets that a student should learn for further work in mathematics. Second, the study of set theory is a special topic in first order predicate logic and/or model theory, both of which are not typically in the early undergraduate curriculum.

More specifically the topics addressed herein include sets, subsets, union and intersection of sets, functions, composition, injectivity, forward image and inverse of a function, set equivalence, and last but certainly not least, the ideas of infinite and finite are introduced. Furthermore, in this project properties and interconnections between these topics are also explored.

A Short History of the Development of Infinite Sets

The notion of using a set to investigate a question is not a new technique. In fact Galileo (1564-1642) in his studies of gravity and motion "...compares two (infinite) sets of velocities..."; these velocities were most likely experimental data [5]. Furthermore "Galileo's argument by comparing two infinitesimal sets provides one of the first such arguments in mathematical history, but one that he uses in other contexts as well" [5, p. 421].

Despite this fact, the rigorization of set theory did not occur until almost three centuries later. It was not until the late 1800's that sets explicitly had a home in the vernacular of the mathematical scene; although at that time sets were referred to as aggregates. The key figure in developing the first main body of results concerning infinite sets is the famous and influential mathematician Goerg Cantor (1845-1918). He discovered through the study of trigonometric series what is now known as the limit point or derived set operator. From this operator he developed the so called transfinite numbers, of which there are two types, ordinals and cardinals [1, p. 86 & p. 111]. Cantor's ideas, so new and ground breaking, were not accepted by all mathematicians. For example Cantor was kept from serving as a professor in Berlin, a quite prestigious position, because the powerful

Kronecker (1823-1891) "... opposed Cantor's theory of infinite sets" [5, p. 732]. Despite Cantor's achievements in founding set theory, his contemporary Dedekind gives quite a wonderful exposition of some of the basic ideas of set theory. Furthermore, as Dedekind points out, Cantor's efforts came from different motivations than did the efforts of Dedekind.

During 1887 in the preface to the first edition of *Was Sind und Was Sollen die Zahlen?* [2], or in English *What Are and What Should Be The (Natural) Numbers?*, upon which the current project is based, Dedekind begins

> In science nothing capable of proof ought to be accepted without proof. Though this demand seems so reasonable yet I cannot regard it as having been met even in the most recent methods of laying the foundations of the simplest science: viz., that part of logic which deals with the theory of numbers. In speaking of arithmetic (algebra, analysis) as a part of logic I mean to imply that I consider the number concept entirely independent of the notions or intuitions of space and time, that I consider it an immediate result from the laws of thought. My answer to the problems propounded in the title of this paper is, then, briefly this: numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things.

In the preface to the second edition of *Was Sind und Was Sollen die Zahlen?* [2], printed in 1893, we can see Dedekind stand behind his work:

> The present memoir after its appearance met with both favorable and unfavorable criticisms; indeed serious faults were charged against it. I have been unable to convince myself of the justice of these charges, and I now issue a new edition of the memoir, which for some time has been out of print, without change, adding only the following note to the first preface.
>
> The property which I have employed as the definition of the infinite system had been pointed out before the appearance of my paper by G. Cantor (*Ein Beitrag zur Mannigfaltigkeitslehre, Crelle's Journal*, Vol 84, 1878), as also by Bolzano (*Paradoxien des Unendlichen*, § 20, 1851). But neither of these authors made the attempt to use this property for the definition of the infinite and upon this foundation to establish with rigorous logic the science of numbers,...

As we can tell Dedekind was quite interested in founding all of mathematics through rigorous logical arguments. In fact this was one motivation for Dedekind to construct the real number system out of the rational number system via the so called method of cuts. Furthermore to find a rigorous definition of the natural numbers is Dedekind's goal in writing *Was Sind und Was Sollen die Zahlen?* Although this project does not follow Dedekind's work to this definition of natural numbers, the interested student is invited to read the whole essay *Was Sind und Was Sollen die Zahlen?* An important issue to raise during the discussion of trying to find a (sound) logical basis for the science of numbers is whether or not this theory of systems of elements admits contradictions or paradoxes.

In 1851, three years after the death of Bolzano (1781-1848), his book *Paradoxien des Unendlichen* was published; a translation of the title is *Paradoxes of the Infinite*. In the early 1900's many paradoxes of sets were being constructed. One of the more famous is the so called Russell's paradox named after Bertrand Russell (1872-1970), in a 1903 publication, but which Russell credits Ernst Zermelo (1871-1953) [5]. Russell's paradox asks whether the set of sets not containing themselves as an element, is an element of itself; the paradox arises since it is an element of itself if and only if it is not an element of itself; check this. This essentially broke open the scene for axiomatic set theory as a more rigorous definition of a set seems to be needed. As this ends our short journey, the development of axiomatic set theory shall not be discussed herein; the interested reader will find [5] is a wonderful book with many references; for set theory see page 807. Let us begin reading Dedekind's considerations on set theory.

The Project

This portion contains work of Dedekind and introduces the notion of a system of elements. The excerpts taken herein are the preliminary set up for Dedekind to explore "What are the natural numbers?" As the original Dedekind is written in German, we read an English translation of his answer to this question [2]. This translation is in some places quite a word for word translation, which is unfortunate because some statements seem phrased like questions. So take care; read carefully and if something sounds a bit funny check the grammar. In that direction, an observant reader will notice the author's translation of the title of Dedekind's essay is not the same as the 1903 translation. A last remark concerning the faithfulness of quoting the source: the symbol that Dedekind uses for the predicate 'is part of' is *not* the symbol that is used herein; although the one herein is similar to the symbol in the original, it is not the same.

Sets and Basic Operations

This section introduces the naive notion of what a set is and two basic operations on sets. Let us begin reading Richard Dedekind.

THE NATURE AND MEANING OF NUMBERS.

I.

SYSTEMS OF ELEMENTS.

1. In what follows I understand by *thing* every object of our thought. In order to be able easily to speak of things, we designate then by symbols, e.g., thing a or of a simply, when we mean the thing denoted by a and not at all the letter a itself. A thing is completely determined by all that can be affirmed or thought concerning it. A thing a is the same as b (identical with b), and b the same as a, when all that can be thought concerning a can also be thought concerning b, and when all that is true of b can also be thought of a. That a and b are only symbols or names for one and the same thing is indicated by the notation $a = b$, and also by $b = a$. If further $b = c$, that is, if c as well as a is a symbol for the thing denoted by b, then is also $a = c$. If the above coincidence of the thing denoted by a with the thing denoted by b does not exist, then are the things a, b said to be different, a is another thing than b, b another thing than a; there is some property belonging to the one that does not belong to the other.

2. It very frequently happens that different things, a, b, c, ... for some reason can be considered from a common point of view, can be associated in the mind, and we say that they form a *system* S; we call the things a, b, c, ... *elements* of the system S, they are *contained* in S; conversely, S *consists* of these elements. Such a system S (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing (1); it is completely determined when with respect to every thing it is determined

whether it is an element of S or not [1]. The system S is hence the same as the system T, in symbols $S = T$, when every element of S is also element of T, and every element of T is also element of S. For uniformity of expression it is advantageous to include also the special case where a system S consists of a *single* (one and only one) element a, i. e., the thing a is element of S, but every thing different from a is not an element of S. On the other hand, we intend here for certain reasons wholly to exclude the empty system which contains no element at all, although for other investigations it may be appropriate to imagine such a system.

3. Definition. A system A is said to be *part* of a system S when every element of A is also element of S. Since this relation between a system A and a system S will occur continually in what follows, we shall express it briefly by the symbol $A \prec S$. The inverse symbol $S \succ A$, by which the same fact might be expressed, for simplicity and clearness I shall wholly avoid, but for lack of a better word I shall sometimes say that S is *whole* of A, by which I mean to express that among the elements of S are found all the elements of A. Since further every element s of a system S by (2) can be itself regarded as a system, we can hereafter employ the notation $s \prec S$.

Exercise 1.**(R)** Discuss the following:

1. Your understanding of what is a 'thing'.
2. How Dedekind uses symbols (for example a and b). Make sure to include remarks concerning the $=$ symbol.
3. Your understanding of 'a system of elements'. Can you see any apparent/inherent problems with this definition?
4. Make up two examples of a system of elements.
5. Describe the difference between an 'element' of a system and a 'part' of a system. Look up the modern *terminology and symbols* used to denote the relations 'is an element of' and 'is a part of', e.g. this information could be found in [3] or [4].
6. What Dedekind says about the 'inverse symbol' for part. The relation Dedekind calls 'is whole of' is commonly called 'contains' in modern terminology.

Note: Modern set theory takes a slightly different point of view than does Dedekind with respect to the empty system or empty set. Today the empty set, commonly denoted $\varnothing$, is included as a set or system of elements.

Exercise 2. Prove $\varnothing$ is a part of any system of elements S.

4. Theorem. $A \prec A$, by reason of (3).

[1] In what manner this determination is brought about, and whether we know a way of deciding upon it, is a matter of indifference for all that follows; the general laws to be developed in no way depend upon it; They hold under all circumstances. I mention this expressly because Kronecker not long ago (*Crelle's Journal*, Vol 99, pp. 334-336) has endeavored to impose certain limitations upon the free formation of concepts in mathematics which I do not believe to be justified; but there seems to be no call to enter upon this matter with more detail until the distinguished mathematician shall have published his reasons for the necessity or merely the expediency of these limitations.

5. Theorem. If $A \prec B$ and $B \prec A$, then $A = B$.

Exercise 3. Prove that the empty system is unique. Hint: show any two empty systems are equal.

The proof follows from (3), (2).

6. Definition. A system A is said to be a *proper* [*echter*] part of S, when A is a part of S, but different from S. According to (5) then S is not a part of A, i.e., there is in S an element which is not an element of A.

7. Theorem. If $A \prec B$ and $B \prec C$, which may be denoted briefly by $A \prec B \prec C$, then is $A \prec C$, and A is certainly a proper part of C, if A is a proper part of B or if B is a proper part of C.

The proof follows from (3), (6).

Exercise 4.**(R)** In the above excerpt Dedekind introduces only one new idea, the notion of a 'proper' part of a set. Suppose that S consists of three elements, say a, b, and c. According to Dedekind's definition, how many proper parts of S are there; list each one of them.

Exercise 5.**(R)** Suppose that a system, S, consists of exactly one element; today we call such a set a *singleton*.

1. Under Dedekind's definition does S have a proper part? Explain clearly why or why not.
2. Under the modern point of view where $\varnothing$ is considered as a set, does S have a proper part? Explain clearly why or why not.

Exercise 6. For each of the theorems 4, 5, and 7:

1. Restate the theorem using modern notation for 'part of'.
2. Reprove the theorem by expanding Dedekind's proof to clearly state how each definition appealed to is being employed.

8. Definition. By the system *compounded* out of any systems A, B, $C, \ldots$ to be denoted by $\mathfrak{M}(A, B, C, \ldots)$ we mean the system whose elements are determined by the following prescription: a thing is considered as element of $\mathfrak{M}(A, B, C, \ldots)$ when and only when it is element of some one of the systems $A, B, C, \ldots$, i.e., when it is element of A, *or* B, *or* C, ... We include also the case where only a single system A exists; then obviously $\mathfrak{M}(A) = A$. We observe further that the system $\mathfrak{M}(A, B, C, \ldots)$ compounded out of A, B, $C, \ldots$ is carefully to be distinguished from the system whose elements are the systems A, B, $C, \ldots$ themselves.

Exercise 7.**(R)** Do the following:

1. Discuss your understanding of the term 'compounded.'
2. Make up an example of a system compounded out of some other systems; make sure to include what these other systems are.
3. Look up the modern *terminology and symbols* used to denote the idea of a system being 'compounded' of other systems; e.g. this information could be found in [3] or [4].
4. Describe the very delicate and important distinction Dedekind is pointing out in the last sentence of Definition 8.

∞∞∞∞∞∞∞∞∞∞

9. Theorem. The systems $A,\ B,\ C, \ldots$ are parts of $\mathfrak{M}\,(A, B, C, \ldots)$.
The proof follows from (8), (3).

10. Theorem. If $A,\ B,\ C, \ldots$ are parts of a system S, then is $\mathfrak{M}\,(A, B, C, \ldots) \prec S$.
The proof follows from (8), (3).

11. Theorem. If P is part of one of the systems $A,\ B,\ C, \ldots$ then is $P \prec \mathfrak{M}\,(A, B, C, \ldots)$.
The proof follows from (9), (7).

12. Theorem. If each of the systems $P,\ Q, \ldots$ is part of one of the systems $A,\ B,\ C, \ldots$ then is $\mathfrak{M}\,(P, Q, \ldots) \prec \mathfrak{M}\,(A, B, C, \ldots)$.
The proof follows from (11), (10).

...

∞∞∞∞∞∞∞∞∞∞

Exercise 8. Restate each of Theorems 9, 10, 11, and 12 using modern terminology and notation.

Exercise 9. Reprove Theorem 10 by expanding Dedekind's proof to clearly state how each definition appealed to is being employed.

Exercise 10. Reprove Theorem 11 by expanding Dedekind's proof to clearly state how each definition appealed to is being employed and to include why Dedekind may use each theorem he utilized.

Exercise 11. Reprove Theorem 12 by expanding Dedekind's proof to include why Dedekind may use each theorem he utilized.

Exercise 12. Suppose the systems $A,\ B,\ C, \ldots$ are given and that $A = \varnothing$. Translate the following into modern notation *and* prove that $\mathfrak{M}\,(A, B, C, \ldots) = \mathfrak{M}\,(B, C, \ldots)$.

∞∞∞∞∞∞∞∞∞∞

17. Definition. A thing g is said to be *common* element of the systems $A,\ B,\ C, \ldots$, if it is contained in each of these systems (that is in A *and* in B *and* in $C \ldots$). Likewise a system T is said to be a *common part* of $A,\ B,\ C, \ldots$ when T is part of each of these systems; and by the name *community* [*Gemeinheit*] of the systems $A,\ B,\ C, \ldots$ we understand the perfectly determinate system $\mathfrak{G}\,(A, B, C, \ldots)$ which consists of all the common elements g of $A,\ B,\ C, \ldots$ and hence is likewise a common part of these systems. We again include the case where only a single system A occurs; then $\mathfrak{G}\,(A)$(is to be put) $= A$. But the case may also occur that the systems $A,\ B,\ C, \ldots$ possess no

common element at all, therefore no common part, no community; they are then called systems *without* common part, and the symbol $\mathfrak{G}(A, B, C, \ldots)$ is meaningless (compare the end of (2)). We shall however almost always in theorems concerning communities leave it to the reader to add in thought the condition of their existence and to discover the proper interpretation of these theorems for the case of non-existence.

Exercise 13.**(R)** Do the following:

1. Discuss your understanding of the terms 'common' element, 'common part', and 'community of systems'.
2. Make up an example of a community of some systems, make sure to include what these systems are.
3. Make up an example of some systems that have no community. Today if two sets fail to have a community, these sets are said to be *disjoint*.
4. Suppose that one is willing to use the empty system; discuss why the symbol $\mathfrak{G}(A, B, C, \ldots)$ must have meaning including the case where there is no common element of $A, B, C \ldots$.
5. Look up the modern *terminology and symbols* used to denote the idea of a system being the 'community' of some systems; e.g. this information could be found in [3] or [4].
6. When including the empty system, does one need to take care that there is a common element of $A, B, C \ldots$ when considering the community of $A, B, C \ldots$; why or why not?

18. Theorem. Every common part of $A,\ B,\ C, \ldots$ is part of $\mathfrak{G}(A, B, C, \ldots)$.

The proof follows from (17).

19. Theorem. Every part of $\mathfrak{G}(A, B, C, \ldots)$ is common part of $A,\ B,\ C, \ldots$.

The proof follows from (17), (7).

20. Theorem. If each of the systems $A,\ B,\ C, \ldots$ is whole (3) of one of the systems $P,\ Q, \ldots$ then is

$$\mathfrak{G}(P, Q, \ldots) \prec \mathfrak{G}(A, B, C, \ldots)$$

Proof. For every element of $\mathfrak{G}(P, Q, \ldots)$ is common element of , therefore also common element of $A,\ B,\ C, \ldots$, which was to be proved.

Exercise 14. Restate each of the Theorems 18, 19 and 20 using modern terminology and notation. This means you **must** consider the empty system in stating the theorem.

Exercise 15. Prove your formulation of Theorem 18 by clearly stating how Definition 17 is being employed. Do not forget to include the case when the community may be empty.

Exercise 16. Prove your formulation of Theorem 19 by clearly stating how Definition 17 and Theorem 7 are utilized while not forgetting to consider the possibility that the community is empty.

Exercise 17. Under the given hypothesis of Theorem 20 prove that a common element of $P,\ Q, \ldots$ is a common element of $A,\ B,\ C, \ldots$

Functions and More

The next portion of reading is broken into two pieces. The first introduces functions and describes how the basic operations defined earlier interact with functions. The second introduces special types of functions for which inverses exist.

Definition, Forward Image, and Composition

In this section we explore how functions interact with the action of taking the community or compounding of systems. Also a basic operation on certain pairs of functions is introduced and a crucial property of this operation is developed.

II.
TRANSFORMATION OF A SYSTEM.

21. Definition.[2] By a *transformation* [*Abbildung*] ϕ of a system S we understand a law according to which every determinate element s of S there *belongs* a determinate thing which is called the *transform* of s and denoted by $\phi(s)$; we say also that $\phi(s)$ *corresponds* to the element s, that $\phi(s)$ *results* or is *produced* from s by the transformation ϕ, that s is *transformed* into $\phi(s)$ by the transformation ϕ. If now T is any part of S, then in the transformation ϕ of S is likewise contained a determinate transformation of T which for the sake of simplicity may be denoted by the same symbol ϕ and consists in this that to every element t of the system T there corresponds the same transform $\phi(t)$, which t possesses as element of S; at the same time the system consisting of all transforms $\phi(t)$ shall be called the transform of T and be denoted by $\phi(T)$, by which also the significance of $\phi(S)$ is defined. ... The simplest transformation of a system is that by which each of its elements is transformed into itself; it will be called the identical transformation of the system. For convenience, in the following theorems (22), (23), (24), which deal with an arbitrary transformation ϕ of an arbitrary system S, we shall denote the transforms of elements s and parts T respectively by s' and T'; in addition we agree that small and capitol italics without accent shall always signify elements and parts of this system S.

Exercise 18.**(R)** Discuss/do the following:

1. Your understanding of the term transformation. Note that in modern terminology S as in the definition above is called the domain of ϕ.
2. What makes two transformations of a system S unequal?
3. Your understanding of the notation $\phi(T)$ where T is a part of S; $\phi(T)$ is called *(forward) ϕ-image of T* in modern terminology.
4. Supposing g designates the identical transformation of some (fixed) system S; fill in the blank using the correct variable(s): $g(s) =$ _________.
5. Make an example, different from the identical transformation, of a transformation of the system S whose three elements are 'red', 'blue' and 'green'.

[2] See Dirichlet's *Vorlesungen über Zahlentheorie*, 3d edition, 1879, §163.

6. Look up the term 'function' in a modern textbook, e.g. [3] or [4], and compare/contrast with Dedekind's definition of transformation; include a remark on contemporary notation verses Dedekind's notation.
7. Describe the significance of the last sentence of Definition 21.

Exercise 19. Prove that for any system S the identical transformation is unique. That is, show that if ϕ and ψ are both identical transformations of S then $\phi = \psi$.

Exercise 20. Let ϕ be a transformation of a system S and let A be a part of S. Show $\phi(A) = \varnothing$ if, and only if, $A = \varnothing$.

∞∞∞∞∞∞∞

22. Theorem.[3] If $A \prec B$ then $A' \prec B'$.

Proof. For every element of A' is the transform of an element contained in A, and therefore also in B. and is therefore element of B', which was to be proved.

∞∞∞∞∞∞∞

Exercise 21. Using modern notation, rewrite the following proof of Theorem 22 by correctly filling in the missing parts from the context (note this proof follows exactly Dedekind's proof):

Assume ________. We want to show ________. Let $y \in \phi(A)$. Then there is ________$\in A$ so that ______$= y$. But by our assumption, we get ______$\in$______. So ______$= y$ for some ______$\in$______; which gives by definition of forward ϕ-image that $y \in$________. Thus from $y \in \phi(A)$ it follows that ________; which was what we wanted to show.

∞∞∞∞∞∞∞

23. Theorem. The transform of $\mathfrak{M}(A, B, C, \ldots)$ is $\mathfrak{M}(A', B', C', \ldots)$.

Proof. If we denote the system $\mathfrak{M}(A, B, C, \ldots)$ which by (10) is likewise part of S by M, then is every element of its transform M' the transform m' of an element m of M; since therefore by (8) m is also element of one of the systems A, B, $C, \ldots$ and consequently m' element of one of the systems A', B', $C', \ldots$, and hence by (8) also element of $\mathfrak{M}(A', B', C', \ldots)$, we have by (3)

$$M' \prec \mathfrak{M}\left(A', B', C', \ldots\right).$$

On the other hand, since A, B, $C, \ldots$ are by (9) parts of M, and hence A', B', $C', \ldots$ by (22) parts of M', we have by (10)

$$\mathfrak{M}\left(A', B', C', \ldots\right) \prec M'.$$

By combination with the above we have by (5) the theorem to be proved

$$M' = \mathfrak{M}\left(A', B', C', \ldots\right).$$

∞∞∞∞∞∞∞

[3] See theorem 27.

Exercise 22. Do the following.

1. Read carefully the proof of Theorem 23. Clearly explain why Dedekind may use each item he referred to (include why for each use of the item since some items are used more than once); e.g. item/Theorem 10 is referred to in both the first sentence and the second to last sentence in the proof by Dedekind.
2. Rewrite Theorem 23 using modern notation for forward image.
3. Using modern notation write your own proof of Theorem 23; you may if you wish use Dedekind's proof as a model for your proof.

24. Theorem.[4] The transform of every common part of $A,\ B,\ C,\ldots$, and therefore that of the community $\mathfrak{G}\left(A,B,C,\ldots\right)$ is part of $\mathfrak{G}\left(A',B',C',\ldots\right)$.

Proof. For by (22) it is common part of $A',\ B',\ C',\ldots$, whence the theorem follows by (18).

Exercise 23. With respect to theorem 24 and Dedekind's proof,

1. What does 'it' refer to in the proof?
2. How and to what is theorem 22 being applied?
3. Why does Theorem 18 complete the proof of Theorem 24?
4. Rewrite Theorem 24 using modern notation for forward image and include the case when there is no common part of $A, B, C, \ldots$
5. Using modern notation write your own proof of Theorem 24.

25. Definition and theorem. If ϕ is a transformation of a system S, and ψ a transformation of the transform $S' = \phi\left(S\right)$, there always results a transformation θ of S, *compounded*[5] out of ϕ and ψ, which consists of this that to every element s of S there corresponds the transform

$$\theta\left(s\right) = \psi\left(s'\right) = \psi\left(\phi\left(s\right)\right),$$

where again we have put $\phi\left(s\right) = s'$. This transformation θ can be denoted briefly by the symbol $\psi \cdot \phi$ or $\psi\phi$, the transform $\theta\left(s\right)$ by $\psi\phi\left(s\right)$ where the order of the symbols ϕ, ψ is to be considered, since in general the symbol $\phi\psi$ has no interpretation and actually has meaning only when $\psi(s') \prec s$. If now χ signifies a transformation of the system $\psi(s') = \psi\phi(s)$ and η the transformation $\chi\psi$ of the system S' compounded out of ψ and χ, then is $\chi\theta(s) = \chi\psi\left(s'\right) = \eta(s') = \eta\phi\left(s\right)$; therefore the compound transformations $\chi\theta$ and $\eta\phi$. In accordance with the meaning of θ and η this theorem can finally be expressed in the form

$$\chi \cdot \psi\phi = \chi\psi \cdot \phi,$$

[4] See theorem 29.

[5] A confusion of this compounding of transformations with that of systems of elements is hardly to be feared.

and this transformation compounded out of $\phi,\ \psi,\ \chi$ can be denoted briefly by $\chi\psi\phi$.

Exercise 24.**(R)** Complete the following:

1. Discuss your understanding of the term 'compounded out of'. Include whether or not the order of compounding is important as well as sufficient conditions for when two functions can be compounded. Is this condition also necessary? Today we refer to 'compounded out of' as "composition of".
2. Make up a system of elements S and two transformations ϕ and ψ as described in the first sentence of Definition 25. What is the resulting transformation compounded out of ϕ and ψ; i.e. what is $\psi \cdot \phi$?
3. Speculate on whether or not $\psi(s') \prec s$ (in the above) should actually be $\psi(S') \prec S$.
4. First observe that the formula $\chi \cdot \psi\phi$ does not utilize parentheses to indicate which composition to do first; describe how Dedekind indicates the order. Second, the equation $\chi \cdot \psi\phi = \chi\psi \cdot \phi$ describes a property of composition known as associativity; discuss how the associative property of 'compounded out of' allows Dedekind to write just $\chi\psi\phi$ without any ambiguity (include this possible ambiguity).

Injective and Inverse Functions

We now explore a slightly more restricted class of functions and explore how the converse of earlier stated theorems behaves for this class of functions. Those functions belonging to this class are called injective and sometimes one-to-one, and each possesses an inverse function. We will let Dedekind introduce the term inverse function or as he would say inverse transformation.

III.
SIMILARITY OF A TRANSFORMATION. SIMILAR SYSTEMS.

26. Definition. A transformation ϕ of a system S is said to be *similar* [*änlich*] or *distinct*, when to different element $a,\ b$ of the system S there always correspond different transforms $a' = \phi(a)$, $b' = \phi(b)$. Since in this case ... from $s' = t'$ we always have $s = t$, then is every element of the system $S' = \phi(S)$ the transform s' of a single, perfectly determinate element s of the system S, and we can therefore set over against the transformation ϕ of S an *inverse* transformation of the system S', to be denoted by $\overline{\phi}$, which consists in this that to every element s' of S' there corresponds the transform $\overline{\phi}(s') = s$, and obviously this transformation is also similar. It is clear that $\overline{\phi}(S') = S$, that further ϕ is the inverse transformation belonging to $\overline{\phi}$ and that the transformation $\overline{\phi}\phi$ compounded out of ϕ and $\overline{\phi}$ by (25) is the identical transformation of S (21). At once we have the following additions to II., retaining the notation there given.

Exercise 25.**(R)** Discuss/do the following:

1. Describe in your own words what it means for a transformation to be 'similar'.
2. Make up two examples of transformations on a (fixed) system S: one that is similar and one that is not.
3. Write the contrapositive of the definition of similar. This is useful to note as sometimes a proof by contraposition is preferable.
4. Look up in a modern textbook, e.g. [3] or [4], 'injective function' and compare/contrast with the definition of 'similar transformation'.
5. Look up in a modern textbook, e.g. [3] or [4], and compare/contrast the definition and notations used there for 'inverse' to the one above used by Dedekind.
6. Describe the difference between the two identical transformations $\overline{\phi}\phi$ and $\phi\overline{\phi}$; can $\overline{\phi}\phi$ and $\phi\overline{\phi}$ be equal? If so, under what conditions; if not, why?

Exercise 26. Prove that if $\overline{\phi}$ is the inverse transformation of ϕ then ϕ is the inverse transformation of $\overline{\phi}$, or symbolically $\overline{\overline{\phi}} = \phi$.

Exercise 27. Let ϕ be a similar transformation of S, and T a part of S. Prove ϕ viewed as a transformation of the system T is also similar.

For the following, Dedekind is assuming the transformation ϕ is similar and is using the notations laid out in section II at the end of definition 21, e.g. s' is $\phi(s)$ and T' is $\phi(T)$.

27. Theorem.[6] If $A' \prec B'$, then $A \prec B$.

Proof. For if a is an element of A then is a' an element of A', therefore also of B' hence $= b'$, where b is an element of B; but since from $a' = b'$ we always have $a = b$, then is every element of A also element of B, which was to be proved.

28. Theorem. If $A' = B'$, then $A = B$.

The proof follows from (27), (4), (5).

29. Theorem.[7] If $G = \mathfrak{G}(A, B, C, \ldots)$ then $G' = \mathfrak{G}(A', B', C', \ldots)$.

Proof. Every element of $\mathfrak{G}(A', B', C', \ldots)$ is certainly contained in S', and is therefore the transform g' of an element g contained in S; but since g' is common element of A', B', $C', \ldots$ then by (27) must g be common element of A, B, $C, \ldots$ therefore also element of G; hence every element of $\mathfrak{G}(A', B', C', \ldots)$ is transform of an element g in G, therefore element of G', i.e., $\mathfrak{G}(A', B', C', \ldots) \prec G'$, and accordingly our theorem follows from (24), (5).

Exercise 28. Suppose that ϕ is a transformation of a system S that is not similar. By choosing appropriate parts, A and B, of S show Theorems 27, 28 and 29 each fail.

Exercise 29. For Theorem 27 what happens if $A' = \varnothing$?

Exercise 30. Write a proof of Theorem 28 using modern notation.

[6] See theorem 22

[7] See theorem 24.

Exercise 31. Restate Theorem 29 using modern notation and forward image notation instead of prime notation; e.g. $\phi(s)$ not s'. Also write your own proof using modern notation, forward image notation, and not forgetting to consider the empty community.

∞∞∞∞∞∞

30. Theorem. The identical transformation of a system is always a similar transformation.

31. Theorem. If ϕ is a similar transformation of S and ψ a similar transformation of $\phi(S)$, then is the transformation $\psi\phi$ of S, compounded of ϕ and ψ, a similar transformation, and the associated inverse transformation $\overline{\psi\phi} = \overline{\phi}\,\overline{\psi}$.

Proof. For two different elements a, b of S correspond different transforms $a' = \phi(a)$, $b' = \phi(b)$, and to these again different transforms $\psi(a') = \psi\phi(a)$, $\psi(b') = \psi\phi(b)$ and therefore $\psi\phi$ is a similar transformation. Besides every element $\psi\phi(s) = \psi(s')$ of the system $\psi\phi(S)$ is transformed by $\overline{\psi}$ into $s' = \phi(s)$ and this by $\overline{\phi}$ into s, therefore $\psi\phi(s)$ is transformed by $\overline{\phi\psi}$ into s, which was to be proved.

∞∞∞∞∞∞

Exercise 32. Write a proof of Theorem 30.

Exercise 33. Explain why every element $\psi\phi(s) = \psi(s')$ of the system $\psi\phi(S)$ is transformed by $\overline{\psi}$ into $s' = \phi(s)$.

Notice that the importance of Theorem 31 is that it tells us that the property of being similar is preserved under composition and how to find the inverse of a composition of two functions in terms of the inverses of those functions composed.

Equivalence of Sets

A very interesting topic is that of equivalent sets. It is through equivalence of sets that Cantor defined the transfinite numbers referred to as cardinals. Intuitively two sets are equivalent if they have the same size or same cardinal number (or cardinality). Let us explore more rigorously.

∞∞∞∞∞∞

32. Definition. The systems R, S are said to be *similar* when there exists such a similar transformation ϕ of S that $\phi(S) = R$, and therefore $\overline{\phi}(R) = S$.

∞∞∞∞∞∞

Exercise 34.**(R)** Do the following:

1. Describe in your own words the term 'similar' when referring to systems of elements (not transformations).
2. Make an example of two (different) systems of elements that are similar; do not forget to point out what transformation establishes these systems are similar. Can there be more than one such transformation; discuss why or why not?
3. Make an example of two systems that are not similar. Explain why these are not similar.

Exercise 35. Prove the last sentence of Definition of 32 that "Obviously ... every system is similar to itself".

Exercise 36. Prove if $R,\ S$ are similar, then $S,\ R$ are similar.

Exercise 37. Prove the even integers and all the integers are similar systems.

33. Theorem. If $R,\ S$ are similar systems, then every system Q similar to R is also similar to S.

Proof. For if $\phi,\ \psi$ are similar transformations of $S,\ R$ such that $\phi(S) = R$, $\psi(R) = Q$, then by (31) $\psi\phi$ is a similar transformation of S such that $\psi\phi(S) = Q$, which was to be proved.

Exercise 38. Without appealing to Theorem 33, prove if R is similar to S and S is similar to Q then R is similar to Q.

Exercise 39.**(I)** The last three exercises showed that the concept 'similar' satisfies properties known as reflexive, symmetric, and transitive, respectively. From these exercises guess a definition for each of the terms reflexive, symmetric, and transitive. Something possessing these properties is called an equivalence relation. **After** making your definitions, look up these terms in a modern textbook e.g. [3] or [4]. How close are your definitions and the ones you looked up?

The next definition begins with the words "We can therefore", indicating that something is at work in the background allowing one to make this definition. In fact, there is a wonderful interplay between equivalence relations and partitions of a system being employed here.

34. Definition. We can therefore separate all systems into *classes* by putting into a determinate class all systems $Q,\ R,\ S, \ldots$, and only those, that are similar to a determinate system R, the *representative* of the class; according to (33) the class is not changed by taking as representative any other system belonging to it.

Exercise 40.**(R)** Describe your understanding of the terms 'classes' and 'representative'.

Exercise 41.**(R)** List all the system(s) that are in the class that has $\varnothing$ as a representative?

35. Theorem. If $R,\ S$ are similar systems, then is every part of S also similar to a part of R, every proper part of S also similar to a proper part of R.

Proof. For if ϕ is a similar transformation of S, $\phi(S) = R$, and $T \prec S$, then by (22) is the system similar to T $\phi(T) \prec R$; if further T is proper part of S, and s an element of S not contained in T then by (27) the element $\phi(s)$ contained in R cannot be contained in $\phi(T)$; hence $\phi(T)$ is proper part of R, which was to be proved.

Exercise 42. What is the phrase "by (22) is the system similar to T $\phi(T) \prec R$;" trying to capture? Note that moving the 'is' to an appropriate place in the sentence will help you answer this question.

Infinite vs. Finite

Since early civilization people have been perplexed and interested by the notion of the infinite. Introduced in this section is the modern definition of the term infinite.

∞∞∞∞∞∞∞∞

IV.
TRANSFORMATION OF A SYSTEM IN ITSELF.

36. Definition. If ϕ is a similar or dissimilar transformation of a system S, and $\phi(S)$ part of a system Z, then ϕ is said to be a transformation of S *in* Z, and we say S is transformed by ϕ in Z. Hence we call ϕ a transformation of the system S *in itself*, when $\phi(S) \prec S$ ...

V.
THE FINITE AND INFINITE.

64. Definition.[8] A system S is said to be *infinite* when it is similar to a proper part of itself (32); in the contrary case S is said to be a *finite* system.

∞∞∞∞∞∞∞∞

Exercise 43.**(R)** Do/answer the following.

1. Describe what it means for a transformation to be 'in Z' as well as 'in itself'. The role that Z plays in this definition is commonly called co-domain or range of the transformation (or function) in modern terminology.
2. Make an example of a system S and a transformation of S in itself.
3. Give an example of an infinite set. Prove it is infinite.
4. Discuss your understanding of the term 'infinite'.
5. Look up in a modern textbook, e.g. [3] or [4], the definition of an infinite set; compare/contrast this definition with Dedekind's definition of infinite.
6. Suppose S is a system such that there is a similar transformation of S in itself, ϕ, with $\phi(S) \neq S$. Is S infinite or finite?

∞∞∞∞∞∞∞∞

65. Theorem. Every system consisting of a single element is finite.
Proof. For such a system possesses no proper part (2), (6).

∞∞∞∞∞∞∞∞

[8] If one does not care to employ the notion of similar systems (32) he must say: S is said to be infinite, when there is a proper part of S (6) in which S can be distinctly (similarly) transformed (26), (36). In this form I submitted the definition of the infinite which forms the core of my whole investigation in September, 1882, to G. Cantor and several years earlier to Schwartz and Weber. All other attempts that have come to my knowledge to distinguish the infinite from the finite seem to me to have met with so little success that I think I may be permitted to forego any criticism of them.

Exercise 44. Since we have agreed on using the empty system we can not accept Dedekind's proof. Why not?

Exercise 45. Prove Theorem 65.

Hint: How many proper parts of a single element system are there and what do you know about systems similar to these proper parts?

Theorem 66, the next theorem, claims the existence of infinite systems. Theorem 66 is crucial for Dedekind in constructing his point of view of the natural numbers. In fact, having an infinite system is the foundation of Dedekind's construction since it is the starting object in his construction of the natural numbers. Before continuing, it is prudent to add a note. Despite some of the inherent flaws of naive set theory that Dedekind's proof glosses over, we have decided to include this theorem in the project for a couple of reasons. First, it sets the stage to ask critical questions about the nature of a system of elements or a set. Second, Dedekind's example of an infinite system is quite interesting, as is the transformation he uses to establish that the system is indeed infinite. An observation a student new to higher level mathematics might make is that Dedekind's proof is much closer to philosophy than to typical calculation-driven mathematics seen previously. Lastly we mention that the existence of infinite systems is now taken as an axiom, not a deducible result, in axiomatic set theory.

66. Theorem. There exist infinite systems.

Proof.[9] My own realm of thoughts, i.e., the totality S of all things, which can be objects of my thought, is infinite. For if s signifies an element of S, then is the thought s', that s can be an object of my thought, itself an element of S. If we regard this as transform $\phi(s)$ of the element s then has the transformation ϕ of S, thus determined, the property that the transform S' is part of S; and S' is certainly proper part of S, because there are elements in S (e.g., my own ego) which are different from such thought s' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that therefore the transformation ϕ is a distinct (similar) transformation (26). Hence S is infinite, which was to be proved.

Exercise 46.**(R)** Comment on this proof. Include the following points: Is the proof satisfactory (to your standards)? Is S (in the proof) actually a system? Is Dedekind's own ego actually a member of S?

67. Theorem. If R, S are similar systems, then is R finite or infinite according as S is finite or infinite.

Proof. If S is infinite, therefore similar to a proper part S' of itself, then if R and S are similar, S' by (33) must be similar to R and by (35) likewise similar to a proper part of R, which therefore by (33) is itself similar to R; therefore R is infinite, which was to be proved.

68. Theorem. Every system S, which possesses an infinite part is likewise infinite; or, in other words, every part of a finite system is finite.

[9] A similar consideration is found in § 13 of the *Paradoxien des Unendlechen* by Bolzano (LeipZig, 1851).

Proof. If T is infinite and there is hence such a similar transformation ψ of T, that $\psi(T)$ is a proper part of T, then, if T is part of S, we can extend this transformation ψ to a transformation ϕ of S in which, if s denotes any element of S, we put $\phi(s) = \psi(s)$ or $\phi(s) = s$ according as s is element of T or not. This transformation ϕ is a similar one; Since further $\psi(T)$ is part of T, because by (7) also part of S, it is clear that also $\phi(S) \prec S$. Since finally $\psi(T)$ is proper part of T there exists in T and therefore also in S, an element t, not contained in $\psi(T) = \phi(T)$; since then the transform $\phi(s)$ of every element s not contained in T is equal to s, and hence is different from t, t cannot be contained in $\phi(S)$; hence $\phi(S)$ is proper part of S and consequently S is infinite, which was to be proved.

∞∞∞∞∞∞

Exercise 47. With regards to Theorem 67:

1. Restate Theorem 67 in the following form: If R and S are similar systems, then ___________ if and only if ______________.
2. Prove your version of Theorem 67.

Exercise 48. With regards to Theorem 68:

1. What allows Dedekind to state "in other words, every part of a finite system is finite"?
2. Prove ϕ, as defined in the proof of Theorem 68, is a similar transformation of S in itself.

∞∞∞∞∞∞

69. Theorem. Every system which is similar to a part of a finite system, is itself finite.
The proof follows from (67), (68).

∞∞∞∞∞∞

Exercise 49. Write a proof of Theorem 69 which clearly indicates how Theorems 68 and 67 are being employed.

∞∞∞∞∞∞

70. Theorem. If a is an element of S, and if the aggregate T of all the elements of S different from a is finite, the is also S finite.

∞∞∞∞∞∞

Dedekind's proof of this result is not included. Instead we indicate a different method of proving Theorem 70.

Exercise 50. Prove Theorem 70 by contraposition. That is, prove if S is infinite, then the aggregate, T, of all elements of S different from a is infinite. One may proceed as follows. Let ϕ establish that S is similar to a proper part of S. First show that if $\phi(T)$ is a part of T then ϕ viewed as a transformation of T will establish that T is infinite. Second show that if $\phi(T)$ is not a part of T then there is ψ, built from ϕ, such that $\psi(T)$ is a part of T. Hint for building ψ: show there are elements b, d in S different from a with $\phi(a) = d$ and $\phi(c) = a$. Now set $\psi(a) = a$ and $\psi(c) = d$ and $\psi(b) = \phi(b)$ for b in S with $b \neq a$ and $b \neq c$.

Exercise 51.**(I)** A result that Dedekind points out after giving quite a rigorous interpretation of the natural numbers is that for any infinite system, S, there is a similar transformation of the (natural) numbers in S.

1. Prove this result by the following method. Let ϕ be a similar transformation of S in itself so that $\phi(S) \neq S$. Pick some element of S that is not an element of $\phi(S)$; send the natural number one to this element. Now $\phi(\phi(S))$ is a proper part of $\phi(S)$, so pick an element in $\phi(S)$ and not in $\phi(\phi(S))$; sent the natural number two to this element. Repeating gives a transformation ψ by sending the natural number as described above.
2. Discuss whether or not the method used above is an allowable method to construct functions.

References

[1] Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, Translation by Phillip E. B. Jourdain, The Open Court Publishing Co., La Salle, Illinois, 1941.

[2] Richard Dedekind, *Essays on the theory of numbers. I: Continuity and irrational numbers. II: The nature and meaning of numbers*, authorized translation by Wooster Woodruff Beman, Dover Publications Inc., New York, 1963.

[3] Susanna S. Epp, *Discrete mathematics with applications–edition 2*, Thomson Learning, Belmont, Ca, 1995.

[4] Paul R. Halmos, *Naive set theory*, Springer-Verlag, New York, 1974, Reprint of the 1960 edition, Undergraduate Texts in Mathematics.

[5] Victor J. Katz, *A history of mathematics, an intro*, HarperCollins College Publishers, New York, 1993.

Notes to Instructor

In this section we share some of our intentions in writing this project as well as some pedagogical motivations that have shaped the project. First it should be mentioned that the project can serve as the primary, complementary or supplementary reading introducing one to sets. If it is used as the primary reading, please note that topics not covered by the project include the power set of a set and set complementation, as well as inverse image of a function; so these concepts need to get addressed by some other means. If the project is used as complementary, then there are times when the students should compare and contrast corresponding material presented in a modern textbook; but it is the author's intention that the students get exposed to Dedekind's description first. Throughout the project the textbooks [3] and [4] are recommended. The former is a more typical undergraduate text and the latter is a more advanced development of set theory; but other textbooks could serve the same purpose. If the project is used as supplementary, then the author recommends reading Dedekind and the corresponding textbook chapters at the same time (or the project first).

Second we mention courses and intended audiences for which the project is suitable. This project is designed mainly for a sophomore or junior level mathematics course that introduces students to naive set theory. Sometimes these courses are entitled 'Discrete Mathematics', 'Finite Mathematics' or 'Introduction to Abstract Mathematics.' For such an audience the author expects that project will take 2 and a half weeks to 3 and a half weeks depending on the number of hours per week the class dedicates to the project and depending on the level of the students. Students with prior exposure to quantified statements may have an advantage when completing the project compared to those who have not. Despite this, students without exposure can get the basics as these students work through the project provided there is sufficient guidance from the instructor. In particular, the connection between the empty set and vacuously true statements is usually a stumbling block for many students.

The project is designed with reading questions, those with (R) after the exercise number, interlaced throughout the project. Usually reading questions follow definitions. The point of these questions is to get students thinking about what they just read. Also these questions form a nice base for a discussion either with the class as a whole or broken into parts. The idea is that the project should create a quite dynamic classroom atmosphere that embraces discussion-led discovery (rather than a lecture-led list of results). Also there are some exercises labeled (I) for investigation question. These questions cover material not presented in the project but that is related or follows naturally from the project; also these questions are meant to serve as a diving board for students to jump into other related topics.

A final remark. One may notice that there are a lot of exercises in this project. An instructor may pick and choose which exercises to assign; note not all exercises need to be assigned. Also, the instructor feeling creative is encouraged to devise other exercises based on the reading of Dedekind. These can be designed with either classroom atmosphere in mind or the content of the material in mind.