An Introduction to Elementary Set Theory

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1 Introduction

In this project we will learn elementary set theory from the original historical sources by two key figures in the development of set theory, Georg Cantor (1845–1918) and Richard Dedekind (1831–1916). We will learn the basic properties of sets. We will also learn how to define the size of a set, and how to compare different sizes of sets. This will lead us to the notions of finite and infinite sets. We will conclude the project by exploring a rather unusual world of infinite sets.

Georg Cantor, the founder of set theory, considered by many as one of the most original minds in the history of mathematics, was born in St. Petersburg, Russia in 1845. His parents, who were of Jewish descent, moved the family to Frankfurt, Germany in 1856. Georg entered the Wiesbaden Gymnasium at the age of 15, and two years later began his university career in Zürich, Switzerland. In 1863 he moved to the University of Berlin, which during Cantor's time was considered the world's leading center of mathematical research. Four years later Cantor received his doctorate under the supervision of the great Karl Weierstrass (1815–1897). In 1869 Cantor obtained an unpaid lecturing post at the University of Halle, which ten years later flourished into a full professorship. However, he never achieved his dream of holding a Chair of Mathematics at Berlin. It is believed that one of the main reasons was the nonacceptance of his theories of infinite sets by the leading mathematicians of that time, most noticeably by Leopold Kronecker (1823–1891), a professor at the University of Berlin and a very influential figure in German mathematics, both mathematically and politically.

Cantor married in 1874 and had two sons and four daughters. Ten years later Georg suffered the first of several mental breakdowns that were to plague him for the rest of his life. Cantor died in 1918 in a mental hospital at Halle. By that time his revolutionary ideas were becoming accepted by some of the leading figures of the new century. One of the greatest mathematicians of the twentieth century, David Hilbert (1862–1943), described Cantor's new mathematics as "the most astonishing product of mathematical thought" [9, p. 359], and claimed that "no one shall ever expel us from the paradise which Cantor has created for us" [9, p. 353]. More on Georg Cantor can be found in [5, 8, 9, 10] and in the literature cited therein.

Julius Wilhelm Richard Dedekind was an important German mathematician, who was also friend to, and an ally of, Cantor. He was born in Braunschweig, Germany in 1831. In

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1848 Richard entered the Collegium Carolinum in Braunschweig, and in 1850 he entered the University of Göttingen—an important German center of mathematics and the home of the great Carl Friedrich Gauss (1777–1855). Dedekind became the last student of Gauss. In 1852 Richard received his doctorate, and spent the next two years at the University of Berlin—the mecca of mathematics of the second half of the nineteenth century. At the University of Berlin, Dedekind and Bernhard Riemann (1826–1866) became contemporaries. They both were awarded the Habilitation in 1854, upon which Dedekind returned to Göttingen to teach as Privatdozent. In Göttingen, Dedekind became close friends with Lejeune Dirichlet (1805–1859). After Dirichlet's death, Dedekind edited Dirichlet's lectures on number theory, which were published in 1863. Dedekind also edited the works of Gauss and Riemann. From 1858 to 1862 Richard taught at the Polytechnic Institute in Zürich. In 1862 his alma mater the Collegium Carolinum was upgraded to a Technische Hochschule (Institute of Technology), and Dedekind returned to his native Braunschweig to teach at the Institute. He spent the rest of his life there. Dedekind retired in 1894, but continued his active mathematical research until his death.

Dedekind is mostly known for his research in algebra and set theory. He was the first to define real numbers by means of cuts of rational numbers. To this day many schools around the globe teach the theory of real numbers based on Dedekind's cuts. Dedekind was the first who introduced the concept of an ideal—a key concept in modern algebra—generalizing the ideal numbers of Ernst Kummer (1810–1893). His contributions to set theory as well as to the study of natural numbers and modular lattices are equally important. In fact, his 1900 paper on modular lattices is considered the first publication in a relatively new branch of mathematics called lattice theory. Dedekind was a well-respected mathematician during his lifetime. He was elected to the Academies of Berlin and Rome as well as to the French Academy of Sciences, and also received honorary doctorates from the universities of Oslo, Zürich, and Braunschweig.

The beginning of Dedekind's friendship with Cantor dates back to 1874, when they first met each other while on holidays at Interlaken, Switzerland. Their friendship and mutual respect lasted until the end of their lives. Dedekind was one of the first who recognized the importance of Cantor's ideas, and became his important ally in promoting set theory.

It is only fitting to study set theory from the writings of Cantor and Dedekind. In this project we will be working with the original historical source by Cantor "Beiträge zur Begründung der transfiniten Mengenlehre" [3] which appeared in 1895, and the original historical source by Dedekind "Was sind und was sollen die Zahlen?" [6] which appeared in 1888. An English translation of Cantor's source is available in [4], and an English translation of Dedekind's source is available in [7].

2 Sets

In the first half of the project our main subject of study will be *sets*. This is how Cantor defined a set:

> Collection into a whole M of definite and separate objects of our intuition or our thought. These objects are called the elements of M.

Examples of sets can be found everywhere around us. For example, we can speak of the set of all living human beings, the set of all cities in the US, the set of all propositions, the set of all prime numbers, and so on. Each living human being is an element of the set of all living human beings. Similarly each prime number is an element of the set of all prime numbers.

If S is a set and s is an element of S, then we write $s \in S$. If it so happens that s is not an element of S, then we write $s \notin S$. If S is the set whose elements are s, t, and u, then we write $S = \{s, t, u\}$. The left brace and right brace visually indicate the "bounds" of the set, while what is written within the bounds indicates the elements of the set. For example, if $S = \{1, 2, 3\}$, then $2 \in S$, but $4 \notin S$.

Sets are determined by their elements. The order in which the elements of a given set are listed does not matter. For example, $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same set. It also does not matter whether some elements of a given set are listed more than once. For instance, $\{1, 2, 2, 2, 3, 3\}$ is still the set $\{1, 2, 3\}$.

Many sets are given a shorthand notation in mathematics as they are used so frequently. A few elementary examples are the sets of natural numbers, integers, rationals, and reals, which are denoted by the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, respectively.

A set may be defined by a property. For instance, the set of all true propositions, the set of all even integers, the set of all odd integers, and so on. Formally, if $P(x)$ is a property, we write $A = \{x \in S : P(x)\}$ to indicate that the set A consists of all elements x of S having the property $P(x)$. The colon : is commonly read as "such that," and is also written as "$|$." So $\{x \in S \,|\, P(x)\}$ is an alternative notation for $\{x \in S : P(x)\}$. For a concrete example, consider $A = \{x \in \mathbb{R} : x^2 = 1\}$. Here the property $P(x)$ is "$x^2 = 1$." Thus, A is the set of all real numbers whose square is one.

Exercise 1 Translate the following to set notation, using predicates to define the respective properties of the variables:

1. The set of all even integers.
2. The set of all odd integers.
3. The set of all prime numbers.

Exercise 2 Give an alternate representation of the following sets, using predicates to define a property if the set cannot be listed explicitly:

1. $\{x \in \mathbb{R} : x^2 = 1\}$.
2. $\{x \in \mathbb{Z} : x > -2 \text{ and } x \leq 3\}$.
3. $\{x \in \mathbb{N} : x = 2y \text{ for some } y \in \mathbb{N}\}$.

2.1 Subset relation

For two sets, we may speak of whether or not one set is contained in the other. This is how Dedekind defines this relation between sets. Note that Dedekind calls sets *systems*.

> A system A is said to be *part* of a system S when every element of A is also an element of S. Since this relation between a system A and a system S will occur continually in what follows, we shall express it briefly by the symbol $A \prec S$.

Modern notation for $A \prec S$ is $A \subseteq S$, and we say that A is a *subset* of S. Formally,

$$A \subseteq S \text{ if and only if for all } x, \text{ if } x \in A, \text{ then } x \in S.$$

When A is not a subset of S, then we write $A \not\subseteq S$.

Exercise 3 Write formally what it means for A not to be a subset of S.

Dedekind goes on to show that the subset relation satisfies the following properties.

Exercise 4

1. Show that $A \subseteq A$.
2. Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Dedekind also defines what it means for A to be a proper part of S.

> A system A is said to be a *proper* part of S, when A is a part of S, but...S is not a part of A, i.e., there is in S an element which is not an element of A.

Nowdays we say that A is a *proper subset* of S, and write $A \subset S$.

Exercise 5

1. Write formally what it means for A to be a proper subset of S.
2. Show that if $A \subset S$, then $A \subseteq S$.
3. Does the converse hold? Justify your answer.
4. Prove that if $A \subset B$ and $B \subset C$, then $A \subset C$.

Although the membership relation $\in$ and the subset relation $\subseteq$ are related to each other, they behave quite differently.

Exercise 6

1. Give an example of a set A such that there is a set B with $B \in A$ but $B \not\subseteq A$.
2. Give an example of a set A such that there is a set B with $B \subseteq A$ but $B \in A$.

2.2 Set equality

One of the primary relations of set theory, as with any mathematical theory, is equality. Intuitively, two sets are equal if they consist of the same elements. When two sets A and B are equal we write $A = B$. Formally,

$$A = B \text{ if and only if for all } x,\ x \in A \text{ if and only if } x \in B.$$

Exercise 7 Prove that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

If two sets A and B are not equal, we write $A \neq B$.

Exercise 8

1. Write formally what it means for two sets not to be equal.
2. Give an example of two sets that are not equal.

Exercise 9

1. For any set A, show that $A = A$.
2. For any two sets A, B, show that $A = B$ implies $B = A$.
3. For any three sets A, B, C, show that $A = B$ and $B = C$ imply $A = C$.

Exercise 10 Consider the sets $A = \{x \in \mathbb{Z} : P(x)\}$ and $B = \{x \in \mathbb{Z} : O(x)\}$, where $P(x)$ is the predicate "x is prime" and $O(x)$ is the predicate "x is odd."

1. Examine A and B with respect to the subset relation. What can you conclude? Justify your answer.
2. Are A and B equal? Justify your answer.

Exercise 11 Consider the sets

$$A = \{x \in \mathbb{Z} : x = 2(y - 2) \text{ for some } y \in \mathbb{Z}\}$$

and

$$B = \{x \in \mathbb{Z} : x = 2z \text{ for some } z \in \mathbb{Z}\}.$$

Are A and B equal? Justify your answer.

2.3 Set operations

The same way we can add, multiply, and subtract numbers, we can create new sets from old ones by taking their sum and product, as well as subtracting one set from another.

The sum of two sets is obtained by combining the elements of the two sets. Thus, for two sets A, B, the *union* of A and B is the set whose elements are all of the elements of A and B. We denote this operation by $\cup$. Formally,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The product of two sets is obtained by taking those and only those elements that the two sets have in common. Thus, for two sets A, B, the *intersection* of A and B is the set consisting of the elements of both A and B. This operation is denoted by $\cap$. Formally,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The difference of two sets consists of the elements of the first set that do not belong to the second set. Thus, for two sets A, B, the difference of A and B is the set consisting of those elements of A that are not in B. This operation is called *set complement* and is denoted by $-$. Formally,

$$A - B = \{x : x \in A \text{ and } x \notin B\}.$$

The notations for the set operations $\cup, \cap, -$, for the membership relation $\in$, and for the subset relation $\subseteq$ that we use today were first introduced by the famous Italian mathematician Giuseppe Peano (1858–1932).[1]

Exercise 12 Let $A = \{2, 3, 5, 7, 11, 13\}$ and $B = \{A, 2, 11, 18\}$.

1. Find $A \cup B$.
2. Find $A \cap B$.
3. Find $A - B$.

Usually the sets that we work with are subsets of some ambient set. For instance, even numbers, odd numbers, and prime numbers are all subsets of the set of natural numbers $\mathbb{N}$. Such an ambient set is referred to as a *universal set* (or a *set of discourse*) and is denoted by U. In other words, a universal set is the underlying set that all the sets under examination are subsets of. We may thus speak of the set difference $U - A$, which is the set of those elements of U that do not belong to A. The set difference $U - A$ is usually denoted by A^c. Formally,

$$A^c = U - A = \{x \in U : x \notin A\}.$$

Exercise 13 Let $A = \{x \in \mathbb{R} : x^2 = 2\}$ and $B = \{x \in \mathbb{R} : x \geq 0\}$.

1. Find $A \cap B$.

[1] Our webpage `http://www.cs.nmsu.edu/historical-projects/` offers a variety of historical projects. For an historical project on Peano see the project [1].

2. Find $A \cup B$.

3. Find $A - B$.

4. For $U = \mathbb{R}$, find A^c and B^c.

5. Find $\mathbb{N} - B$.

2.4 Empty set

As we just saw, the set operations may yield a set containing no elements.

Exercise 14

1. Let A be any set and let E be a set containing no elements. Prove that $E \subseteq A$.

2. Conclude that there is a unique set containing no elements.

We call the set containing no elements the *empty set* (or *null set*) and denote it by $\emptyset$.

Exercise 15 Write a formal definition of the empty set.

Exercise 16 Consider the following sets:

1. $A = \{x \in \mathbb{R} : x^2 + 1 = 0\}$

2. $B = \{x \in \mathbb{N} : x^2 = 2\}$

3. $C = \{x \in \mathbb{Q} : x^2 = 2\}$

4. $D = \{x, y \in \mathbb{N} : x \neq y \text{ and } x^2 + y^2 = 2\}$

Determine an equivalent representation for each of these sets. Justify your claims.

2.5 Set identities

There are a number of set identities that the set operations of union, intersection, and set difference satisfy. They are very useful in calculations with sets. Below we give a table of such set identities, where U is a universal set and A, B, and C are subsets of U.

Commutative Laws:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative Laws:	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive Laws:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotent Laws:	$A \cup A = A$	$A \cap A = A$
Absorption Laws:	$A \cap (A \cup B) = A$	$A \cup (A \cap B) = A$
Identity Laws:	$A \cup \emptyset = A$	$A \cap U = A$
Universal Bound Laws:	$A \cup U = U$	$A \cap \emptyset = \emptyset$
DeMorgan's Laws:	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$
Complement Laws:	$A \cup A^c = U$	$A \cap A^c = \emptyset$
Complements of U and $\emptyset$:	$U^c = \emptyset$	$\emptyset^c = U$
Double Complement Law:	$(A^c)^c = A$	
Set Difference Law:	$A - B = A \cap B^c$	

Exercise 17

1. Prove the commutative laws.
2. Prove the associative laws.
3. Prove the idempotent laws.
4. Prove the identity laws.
5. Prove the universal bound laws.

Exercise 18

1. Prove the complement laws.
2. Prove the complement of U and $\emptyset$ laws.
3. Prove the double complement law.
4. Prove the difference law.

Exercise 19

1. Prove the absorbtion laws.
2. Prove DeMorgan's laws.
3. Prove the distributive laws.

Exercise 20 Prove the following using only set identities:

1. $(A \cup B) - C = (A - C) \cup (B - C)$.
2. $(A \cup B) - (C - A) = A \cup (B - C)$.
3. $A \cap (((B \cup C^c) \cup (D \cap E^c)) \cap ((B \cup B^c) \cap A^c)) = \emptyset$.

2.6 Powerset

Let A be a set. Then we may speak of the set of all subsets of A. This is yet another operation on sets, which is of great importance. We call the set of all subsets of A the *powerset* of A and denote it by $P(A)$. Formally,

$$P(A) = \{B : B \subseteq A\}.$$

For example, if $A = \{1, 2\}$, then the subsets of A are $\emptyset, \{1\}, \{2\}$, and A. Therefore, $P(A) = \{\emptyset, \{1\}, \{2\}, A\}$.

Exercise 21

1. Determine $P(\emptyset)$.

2. Determine $P(\{1\})$.
3. Determine $P(\{1, 2, 3\})$.

Exercise 22

1. Calculate $P(\{\emptyset\})$.
2. Calculate $P(\{\emptyset, \{\emptyset\}\})$.
3. Calculate $P(\{\{\emptyset\}\})$.
4. Calculate $P(P(\emptyset))$.
5. Calculate $P(P(\{\emptyset\}))$.

2.7 Cartesian products

We recall that the order in which we list the elements of a given set does not matter. Nevertheless, it is common practice in mathematics (and other disciplines) to speak about ordered pairs. Thus, we define the *Cartesian product* of two sets A, B to be the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. We denote the Cartesian product of A and B by $A \times B$. Formally,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Exercise 23 Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Determine $A \times B$ and $B \times A$.

Exercise 24

1. Let A consist of 4 elements and B consist of 5 elements. How many elements are in $A \times B$? Justify your answer.
2. More generally, let A consist of n elements and B consist of m elements. How many elements are in $A \times B$? Justify your answer.

2.8 Russell's paradox

As we saw earlier in the project, sets are determined by properties. Since, formally speaking, properties are predicates, it appears that all sets can be obtained by means of predicates. Therefore, the whole of set theory and hence the whole of mathematics appears to be derivable from the general principles of logic. This was the grand plan of the great German mathematician, philosopher, and one of the founders of modern logic Gottlob Frege (1848–1925). His plan is known as *logicism*. Unfortunately, soon after Frege published his program, the famous British philosopher, mathematician, and antiwar activist Bertrand Russell (1872–1970) found a fatal flaw in Frege's arguments. This became known as *Russell's paradox*.[2] We conclude the first half of the project by examining closely Russell's paradox. Russell's argument goes as follows (see, e.g., [12, p. 2]):

[2]For historical projects on Frege and Russell see the projects [11, 2] on our webpage `http://www.cs.nmsu.edu/historical-projects/`.

By a set, we mean any collection of objects, e.g., the set of all even integers, the set of all saxophone players in Brooklyn, etc. The objects which make up a set are called its members. Sets may themselves be members of sets, e.g., the set of all sets of integers has sets as its members. Most sets are not members of themselves; the set of cats, for example, is not a member of itself, because the set of cats is not a cat. However, there may be sets which do belong to themselves, e.g., the set of all sets. Now, consider the set A of all those sets X such that X is not a member of X. Clearly, by definition, A is a member of A if any only if A is not a member of A. So, if A is a member of A, then A is also not a member A; and if A is not a member of A, then A is a member of A. In any case, A is a member of A and is not a member of A.

Let A be the set of all those sets that are not members of themselves.

Exercise 25 Give a formal definition of A.

The question we will examine is whether A is a member of itself.

Exercise 26

1. First assume that $A \in A$ and conclude that $A \notin A$. Justify your argument.

2. Next assume that $A \notin A$ and conclude that $A \in A$. Justify your argument.

3. What can you conclude from (1) and (2)? Explain.

4. Discuss why Russell's paradox contradicts Frege's program.

5. How would you resolve the situation? Explain.

3 Functions, one-to-one correspondences, and cardinal numbers

So far in this project we have studied such basic relations as membership, subset, and equality relations. We have also studied basic operations on sets such as union, intersection, set difference, powerset, and Cartesian product. Our next goal is to discuss the "size" of sets. We have already encountered sets of large and small sizes. Some sets that we have encountered were finite and some were infinite. Our next goal is to formalize the concept of the size of a set. As we will see, this can be done by means of *functions*—one of the key concepts in mathematics.

We will learn about functions, one-to-one and onto functions, one-to-one correspondences, and how they allow us to formalize the concept of the size of a set. A formal definition of the size of a set is that of the *cardinality* of a set. We will discuss what it means for two sets to be equivalent, and study how to compare the sizes of different sets. We will introduce countable sets and show that many sets are countable, including the set of integers and the set of rational numbers. We will also discuss Cantor's diagonalization method which allows us to show that not every infinite set is countable. This will show that infinite sets may have different sizes. In particular, we will show that the set of real numbers is not countable. We will also examine the cardinal number $\aleph_0$, the first in the hierarchy of infinite cardinal numbers, and obtain a method that allows us to create infinitely many infinite cardinal numbers.

3.1 Functions

Let A and B be sets. Speaking informally, a *function* from A to be B is a rule associating with each element of A *one and only one* element of B. This is how Dedekind defines a function. Note that he refers to functions as transformations.

> By a *transformation* ϕ of a system S we understand a law according to which to every determinate element s of S there *belongs* a determinate thing which is called the *transform* of s and denoted by $\phi(s)$; we say also that $\phi(s)$ *corresponds* to the element s, that $\phi(s)$ *results* or is *produced* from s by the transformation ϕ, that s is *transformed* into $\phi(s)$ by the transformation ϕ.

When there is a function f from A to be B, we write $f : A \to B$. The set A is referred to as the *domain* of f, and the set B is referred to as the *codomain* of f. Since the function f associates with each $a \in A$ a unique $b \in B$, we say that f *sends* a *to* b and write $f(a) = b$.

Formally speaking, a function $f : A \to B$ is the set of ordered pairs (a, b), where $a \in A$, $b \in B$, and f sends a to b. Note that from the definition of a function, we cannot have two ordered pairs (a, b) and (a, c) with $a \neq c$. Thus, we can think of functions from A to B as subsets F of $A \times B$ which satisfy the following property: For each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in F$.

Exercise 27 Are the following relations functions? Justify your answer.

1. $f(x) = x^2$ with domain and codomain $\mathbb{R}$.
2. $g(x) = 2x + 1$ with domain and codomain $\mathbb{Q}$.
3. $h(x) = \pm x$ with domain and codomain $\mathbb{Z}$.
4. $u(x) = \sqrt{x}$ with domain and codomain $\mathbb{N}$.

Exercise 28 Write each of the following functions as a set of ordered pairs.

1. $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \cos(x)$.
2. $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 300x$.
3. $h : \mathbb{R}^+ \to \mathbb{R}$ defined by $h(x) = \ln(x)$.

For a function $f : A \to B$, we call the set of all values of f the *range* or *image* of f. Thus, the image of f is the set

$$\mathrm{Im}(f) = \{b \in B : b = f(a) \text{ for some } a \in A\}.$$

Exercise 29 Consider the function $f = \{(1, 2), (2, 3), (3, 3), (4, 5), (5, -1), (6, 2)\}$. Identify the domain, codomain, and image of f.

3.2 Images and inverse images

Let $f : A \to B$ be a function, $S \subseteq A$, and $T \subseteq B$. The image of S with respect to f is the set of those elements of B which the elements of S are sent to. Therefore, the *image* of S with respect to f is the set

$$f(S) = \{f(s) : s \in S\}.$$

On the other hand, the inverse image of T with respect to f is the set of those elements of A that are sent to some element of T. Thus, the *inverse image* of T with respect to f is the set

$$f^{-1}(T) = \{a \in A : f(a) \in T\}.$$

Exercise 30 For each of the following functions determine the image of $S = \{x \in \mathbb{R} : 9 \leq x^2\}$.

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$.
2. $g : \mathbb{R} \to \mathbb{R}^+$ defined by $g(x) = e^x$. Here and below $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.
3. $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x - 9$.

Exercise 31 For each of the following functions determine the inverse image of $T = \{x \in \mathbb{R} : 0 \leq x^2 - 25\}$.

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x^3$.
2. $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \ln(x)$.
3. $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x - 9$.

3.3 When are two functions equal?

Let f and g be two functions from A to B. We say that f *equals* g and write $f = g$ if $f(a) = g(a)$ for each $a \in A$.

Exercise 32 Determine whether each of the following pairs of functions are equal. Justify your answer.

1. $f = \{(a, b) : a \in \mathbb{Z} \text{ and } b = 2a^2 - a\}$ and $g = \{(x, y) : x \in \mathbb{Z} \text{ and } y = x(2x - 1)\}$.
2. $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $f(x) = \dfrac{1}{x}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \dfrac{1}{x}$.

3.4 Composition

Given two functions $f : A \to B$ and $g : B \to C$, we can produce a new function $h : A \to C$ by *composing* f and g. This is how Dedekind defines the composition of two functions.

> If ϕ is a transformation of a system S, and ψ a transformation of the transform $S' = \phi(S)$, there always results a transformation θ of S, *compounded* out of ϕ and ψ, which consists of this that to every element s of S there corresponds the transform
>
> $$\theta(s) = \psi(s') = \psi(\phi(s)),$$
>
> where again we have put $\phi(s) = s'$. This transformation θ can be denoted briefly by the symbol $\psi \cdot \phi$ or $\psi\phi$...

Thus, if $f : A \to B$ and $g : B \to C$ are two functions, then their *composition* is defined as the function $h : A \to C$ such that $h(a) = g(f(a))$ for each $a \in A$. We denote the composition of f and g by $g \circ f$.

Exercise 33 For each of the following three functions, defined over the appropriate subsets of $\mathbb{R}$, determine $h \circ (g \circ f)$ and $(h \circ g) \circ f$. Are they equal?

1. $f(x) = \dfrac{1}{x}$, $g(x) = \ln(3x^2 - e^x)$, $h(x) = x^3$.
2. $f(x) = 3x - 1$, $g(x) = \ln(e^x)$, $h(x) = \dfrac{1}{3x-1}$.
3. $f(x) = x^{\ln(x)}$, $g(x) = x^4 - 12x$, $h(x) = \dfrac{x}{x^3 - x^2}$.

Exercise 34 In this exercise we generalize the results of Exercise 33 and show that $h \circ (g \circ f) = (h \circ g) \circ f$ for any functions $f : A \to B$, $g : B \to C$, and $h : C \to D$.

1. State what you need to show to conclude that $h \circ (g \circ f) = (h \circ g) \circ f$.
2. Consider now some $a \in A$. What can you say about $f(a)$? What can you say about $(g \circ f)(a)$? What can you say about $h((g \circ f)(a))$?
3. Consider the same $a \in A$ as before. What can you say about $(h \circ g)(f(a))$?
4. Use your solutions to (1)–(3) to conclude that $h \circ (g \circ f) = (h \circ g) \circ f$.

Let A be an arbitrary set. The *identity function* $i_A : A \to A$ is the function defined by $i_A(a) = a$ for each $a \in A$. In other words, the identity function i_A sends each $a \in A$ to itself.

Exercise 35 Let $f : A \to B$ be a function.

1. Show that for the identity function i_A on A we have $f \circ i_A = f$.
2. Show that for the identity function i_B on B we have $i_B \circ f = f$.

3.5 One-to-one and onto functions

Let $f : A \to B$ be a function. As we saw, it could happen that f sends several elements of A to the same element of B. We say that f is a *one-to-one function* (or an *injective function*) if f sends each element of A to a unique element of B. Thus, f is one-to-one if for each $a_1, a_2 \in A$, from $f(a_1) = f(a_2)$ it follows that $a_1 = a_2$.

Exercise 36 Let $f : A \to B$ be a function. Show that the following two conditions are equivalent:

1. f is one-to-one.

2. For each $a_1, a_2 \in A$, whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

In fact, both of these conditions are equivalent to a third condition stating that $S = f^{-1}(f(S))$ for each $S \subseteq A$. But this is a little more challenging to prove. (Try!)

For a function $f : A \to B$ it could happen that the image of f is a proper subset of the codomain of f. We say that f is an *onto function* (or a *surjective function*) if the image of f equals the codomain of f. Thus, f is onto if for each $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$. One can show that f is onto if and only if $T = f(f^{-1}(T))$ for each $T \subseteq B$. This is a little more challenging to prove. (Give it a try!)

3.6 One-to-one correspondences and set equivalence

Let $f : A \to B$ be a function. If it happens that f is both one-to-one and onto, then we say that f is a *one-to-one correspondence* (or a *bijection*) between A and B.

Exercise 37 Consider the following two functions:

1. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 15$.

2. $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 15x^3$.

Prove that both f and g are one-to-one correspondences.

Let $f : A \to B$ be a one-to-one correspondence. Then to each $a \in A$ there is a unique $b \in B$ such that $f(a) = b$. We define $f^{-1} : B \to A$ by

$$f^{-1}(b) = \text{the unique } a \text{ such that } f(a) = b.$$

Exercise 38 Let $f : A \to B$ be a one-to-one correspondence.

1. Prove that f^{-1} is a function.

2. Prove that f^{-1} is one-to-one.

3. Prove that f^{-1} is onto.

4. Conclude that $f^{-1} : B \to A$ is a one-to-one correspondence.

Exercise 39 Let $f : A \to B$ be a one-to-one correspondence. By Exercise 38, $f^{-1} : B \to A$ is also a one-to-one correspondence.

1. Prove that $f^{-1} \circ f = i_A$.
2. Prove that $f \circ f^{-1} = i_B$.

Informally speaking, if $f : A \to B$ is a one-to-one function, then as each element in A is sent to exactly one element in B, the size of B is at least as large as the size of A. On the other hand, if f is onto, then as each element in B has at least one element in A that it is the image of, the size of B is no greater than the size of A. Thus, one-to-one correspondences provide us with a means to compare the sizes of sets. This key observation of Cantor led him to the notion of two sets being equivalent. Let us read how Cantor defines that two sets are equivalent.

> We say that two aggregates M and N are "equivalent," in signs
>
> $$M \sim N \quad \text{or} \quad N \sim M,$$
>
> if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.

Next Cantor states that each set is equivalent to itself, and that if a set is equivalent to two other sets, then the two sets are also equivalent.

> Every aggregate is equivalent to itself:
>
> $$M \sim M.$$
>
> If two aggregates are equivalent to a third, they are equivalent to one another, that is to say:
>
> $$\text{from} \quad M \sim P \quad \text{and} \quad N \sim P \quad \text{follows} \quad M \sim N.$$

Exercise 40 Prove the above two claims of Cantor.

3.7 Cardinality of a set, cardinal numbers

As we saw in the previous section, two sets A and B having the same size can be formalized by saying that the sets A and B are equivalent. All equivalent sets have the same size. One of the key breakthroughs of Cantor was to introduce new numbers, which he called *cardinal numbers*, measuring the size of sets. Let us read how Cantor defined the cardinality of a set.

> Every aggregate M has a definite "power," which we also call its "cardinal number."
>
> We will call by the name "power" or "cardinal number" of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given.
>
> We denote the result of this double act of abstraction, the cardinal number or power of M, by
>
> $$\overline{\overline{M}}.$$

Exercise 41 What do you think Cantor means by "cardinal number"? Why? Given a set A consisting of ten round marbles, each of a different color, what is $\overline{\overline{A}}$?

In the next excerpt, Cantor connects the two key notions, that of cardinality and that of set equivalence.

> Of fundamental importance is the theorem that two aggregates M and N have the same cardinal number if, and only if, they are equivalent: thus,
>
> $$\text{from } M \sim N, \text{ we get } \overline{\overline{M}} = \overline{\overline{N}},$$
>
> and
>
> $$\text{from } \overline{\overline{M}} = \overline{\overline{N}}, \text{ we get } M \sim N.$$
>
> Thus the equivalence of aggregates forms the necessary and sufficient condition for the equality of their cardinal numbers.

Exercise 42 Explain in your own words what Cantor means in the above.

Exercise 43 Let S be the set of all perfect squares

$$\{0, 1,\ 4,\ 9,\ 16,\ 25,\ \ldots\}.$$

From Cantor's statement above, do S and $\mathbb{N}$ have the same cardinality? Justify your answer.

Exercise 44 Do $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality? Justify your answer.

Exercise 45 Do $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality? Justify your answer. (Hint: Draw a picture of $\mathbb{N} \times \mathbb{N}$. Can you label each element of $\mathbb{N} \times \mathbb{N}$ by a unique natural number?)

Exercise 46 Does $\mathbb{Q}$ have the same cardinality as $\mathbb{N}$? Justify your answer. (Hint: Establish a one-to-one correspondence between $\mathbb{Q}$ and a subset of $\mathbb{Z} \times (\mathbb{N} - \{0\})$ and modify your solution to Exercise 45.)

3.8 Ordering of cardinal numbers

Some sets have larger size than others. Since cardinal numbers measure the size of sets, it is natural to speak about one cardinal number being less than the other. This is exactly what Cantor does in the next excerpt.

> If for two aggregates M and N with the cardinal numbers $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$, both the conditions:
>
> (a) There is no part of M which is equivalent to N,
>
> (b) There is a part N_1 of N, such that $N_1 \sim M$,

are fulfilled, it is obvious that these conditions still hold if in them M and N are replaced by two equivalent aggregates M' and N'. Thus they express a definite relation of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ to one another.

Further, the equivalence of M and N, and thus the equality of $\mathfrak{a}$ and $\mathfrak{b}$, is excluded; for if we had $M \sim N$, we would have, because $N_1 \sim M$, the equivalence $N_1 \sim N$, and then, because $M \sim N$, there would exist a part M_1 of M such that $M_1 \sim M$, and therefore we should have $M_1 \sim N$; and this contradicts the condition (a).

Thirdly, the relation of $\mathfrak{a}$ to $\mathfrak{b}$ is such that it makes impossible the same relation of $\mathfrak{b}$ to $\mathfrak{a}$; for if in (a) and (b) the parts played by M and N are interchanged, two conditions arise which are contradictory to the former ones.

We express the relation of $\mathfrak{a}$ to $\mathfrak{b}$ characterized by (a) and (b) by saying: $\mathfrak{a}$ is "less" than $\mathfrak{b}$ or $\mathfrak{b}$ is "greater" than $\mathfrak{a}$; in signs

$$\mathfrak{a} < \mathfrak{b} \text{ or } \mathfrak{b} > \mathfrak{a}.$$

Exercise 47 Describe in your own words what it means for two cardinals $\mathfrak{a} = \overline{\overline{M}}$ and $\mathfrak{b} = \overline{\overline{N}}$ to be in the relation $\mathfrak{a} < \mathfrak{b}$.

Cantor states the following:

We can easily prove that,

$$\text{if } \mathfrak{a} < \mathfrak{b} \text{ and } \mathfrak{b} < \mathfrak{c}, \text{ then we always have } \mathfrak{a} < \mathfrak{c}.$$

Exercise 48 Prove the above claim of Cantor.

Exercise 49

1. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers. Modify Cantor's definition of $\mathfrak{a} < \mathfrak{b}$ to define $\mathfrak{a} \leq \mathfrak{b}$.

2. Prove that $\mathfrak{a} \leq \mathfrak{a}$.

3. Prove that if $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{c}$, then $\mathfrak{a} \leq \mathfrak{c}$.

4. Do you think that $\mathfrak{a} \leq \mathfrak{b}$ and $\mathfrak{b} \leq \mathfrak{a}$ imply $\mathfrak{a} = \mathfrak{b}$? Explain your reasoning. (Hint: This is not as trivial as it might look.)

3.9 Finite and infinite sets

Now that we have a good understanding of cardinal numbers and how they compare to each other, we can speak formally about finite and infinite sets. Intuitively a set is finite if it consists of finitely many elements and it is infinite otherwise. Formally, a set A is *finite* or has *finite cardinality* if there is $n \in \mathbb{N}$ such that A is equivalent to the set $\{0, \ldots, n-1\} \subset \mathbb{N}$. On the other hand, A is *infinite* or has *infinite cardinality* if A is not equivalent to any finite subset of $\mathbb{N}$.

This is how Dedekind defines finite and infinite sets. Note that Dedekind calls equivalent sets *similar*.

> A system S is said to be *infinite* when it is similar to a proper part of itself...in the contrary case S is said to be a *finite* system.

Exercise 50

1. Explain Dedekind's definition in your own words.
2. According to Dedekind's definition, are $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ infinite sets? Explain.
3. Do you think that Dedekind's definition is equivalent to the one given above? Give your reasoning. (Hint: This is not entirely trivial.)

3.10 Countable sets

In the rest of the project we will concentrate on infinite sets and infinite cardinal numbers. Our first goal is to identify a special infinite cardinal number that Cantor calls *aleph-zero*.

> The first example of a transfinite aggregate is given by the totality of finite cardinal numbers ν; we call its cardinal number "Aleph-zero," and denote it by $\aleph_0$;

Note that $\aleph_0$ is the first letter of the Hebrew alphabet. In modern terminology, a set whose cardinal number is $\aleph_0$ is called *countably infinite*.

Exercise 51 What symbol is used today to denote the "totality of finite cardinal numbers ν"? Explain.

Cantor claims that $\aleph_0$ is greater than any finite cardinal number:

> The number $\aleph_0$ is greater than any finite number μ:
>
> $$\aleph_0 > \mu.$$

Exercise 52 Prove the above claim of Cantor.

Sets which are either finite or countably infinite are called *countable sets*. There are many examples of countable sets. For instance, finite sets, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all examples of countable sets. The next natural question is whether there exist uncountable sets, and if so, how to construct them.

3.11 Uncountable sets and higher levels of infinity

First we show that $\aleph_0$ is the smallest among infinite cardinal numbers. This is exactly what Cantor does in the next excerpt.

> ...$\aleph_0$ is the least transfinite cardinal number. If $\mathfrak{a}$ is any transfinite cardinal number different from $\aleph_0$, then
>
> $$\aleph_0 < \mathfrak{a}.$$

Exercise 53 Prove the above claim of Cantor. (Hint: Let $\mathfrak{a} = \overline{\overline{A}}$. Can you define a one-to-one function from $\mathbb{N}$ to A? For this you will need to choose some elements from A.)

Exercise 54 Let $[0, 1]$ denote the set of all real numbers between 0 and 1. Show that $\aleph_0 < \overline{\overline{[0, 1]}}$. We outline what is now known as *Cantor's diagonalization method* as one way to prove this. Represent real numbers in $[0, 1]$ as infinite decimals (which do not end in infinitely repeating 9's). Assume that $\mathbb{N} \sim [0, 1]$. Then to each infinite decimal one can assign a unique natural number, so the infinite decimals can be enumerated as follows:

$$\begin{array}{c} .a_{11}a_{12}\ldots a_{1n}\ldots \\ .a_{21}a_{22}\ldots a_{2n}\ldots \\ \vdots \\ .a_{n1}a_{n2}\ldots a_{nn}\ldots \\ \vdots \end{array}$$

Can you construct an infinite decimal $.b_1b_2\ldots b_n\ldots$ such that $a_{nn} \neq b_n$ for each positive n? What can you conclude from this?

Exercise 55

1. Is $\overline{\overline{[0, 1]}}$ strictly greater than $\aleph_0$? Justify your answer.
2. Is $\overline{\overline{\mathbb{R}}}$ strictly greater than $\aleph_0$? Justify your answer.

Now that we know that not every infinite set is countable, it is natural to ask whether we can create larger and larger infinite sets. The answer to this question is again *yes*. The proof of this important fact is based on the *generalized version of Cantor's diagonalization method*.

Exercise 56 Let A be a set and $P(A)$ be the powerset of A. Prove the following claim of Cantor:

$$\overline{\overline{P(A)}} > \overline{\overline{A}}.$$

Hint: Employ Cantor's generalized diagonalization method. Assume that $A \sim P(A)$. Then there is a one-one correspondence $f : A \to P(A)$. Consider the set $B = \{a \in A : a \notin f(a)\}$. Can you deduce that $B \subseteq A$ is not in the range of f? Does this imply a contradiction?

Exercise 57 Using the previous exercise, describe an infinite increasing sequence of infinite cardinal numbers.

Notes to the Instructor

This project is based on the authors' experience in teaching discrete mathematics from primary historical sources. It was designed to serve the needs of college freshmen and sophomores who are meeting mathematical proofs for the first time. Although for some exercises a basic familiarity with first-year calculus is useful, no specific prerequisites are assumed.

The project has a large variety of exercises. Instructors can pick and choose which exercises to assign, depending on what parts of the project they will cover. In our experience, students have little to no difficulty understanding the material presented in the first half of the project. However, the material pertaining to functions in general, and to images and inverse images in particular, requires instructor guidance. It is advisable to have a detailed class discussion on some of the excerpts of Cantor and Dedekind about set equivalence and the cardinality of a set, as well as about countable and uncountable sets.

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