Peano Arithmetic

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1 Introduction

In this project we will learn the first-order theory of arithmetic, known as *Peano Arithmetic*. The formal development of arithmetic goes all the way back to ancient Greek mathematics. But the modern theory of arithmetic was developed only in the second half of the nineteenth century with the work of Hermann Grassmann (1809–1877), Richard Dedekind (1833–1916), and Gottlob Frege (1848–1925). However, it was not until Giuseppe Peano's (1858–1932) treatise *"Arithmetices principia, nova methodo exposita"* (The principles of arithmetic, presented by a new method) that the axiomatic theory of arithmetic was devised as we know it today.

We note that Grassmann's treatise [6] was the earliest, and appeared over 20 years before the work of Frege, Dedekind, and Peano. On the other hand, the treatises of Frege [5], Dedekind [3], and Peano [9] appeared within five years of each other, with Frege's publication in 1884 being the earliest, and Peano's in 1889 the latest. Although the work of Frege, Dedekind, and Peano have a lot in common, it follows from their own testaments that they were not aware of each others work at the time of completion of their own.

We will learn the first-order theory of arithmetic based on the original source of Peano [9]. We will examine Peano's celebrated postulates for arithmetic, paying special attention to his axiomatization of the principle of mathematical induction. We will also learn how to axiomatize addition and multiplication and how to derive their basic properties such as commutativity, associativity, and distributivity of multiplication over addition.

At the end of the project we will examine the strength of the first-order axiomatization of the principle of mathematical induction. We will study Dedekind's Theorem, which proves that the natural numbers are unique up to isomorphism provided that we can use the full strength of the principle of mathematical induction. We will also examine Skolem's Theorem, which shows that the first-order axiomatization of arithmetic (which cannot express the full strength of the principle of mathematical induction) has countable models which are not isomorphic to the natural numbers.

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2 Giuseppe Peano

Giuseppe Peano was born on 27 August of 1858 near Turin, in the village of Spinetta. He went to village school first in Spinetta and then in nearby Cuneo. In the early seventies he continued his studies in Turin. In 1876 Giuseppe enrolled in the University of Turin, from which he graduated in 1880 with 'high honors.'

Peano spent his entire academic career at the University of Turin, starting as an assistant in 1880, and steadily progressing through the ranks all the way until he reached the rank of a full chair, which he held from 1895 until his death in 1932. In 1886–1901 he also held a position of professor at the Royal Military Academy of Turin.

Peano's main contributions were in analysis, axiomatization of mathematics, and most importantly in mathematical logic. In fact, in the late nineteenth and early twentieth centuries he was considered the leading figure in mathematical logic alongside Frege and Bertrand Russell (1872–1970).

In analysis Peano is best known for his space-filling curve, which bears his name. But he also contributed to differential equations and measure theory. In fact, Henri Lebesgue (1875–1941) acknowledged Peano's influence on his own groundbreaking research.

Peano was a great proponent of Grassmann's revolutionary development of linear algebra. Actually, Peano was one of the first who realized the importance of Grassmann's work. His book [8] gives the first axiomatic development of vector spaces.

It is remarkable and not well known that Peano was the inventor of the symbol "$\in$" that we commonly use today to denote the set membership relation. But perhaps the most widely known of Peano's accomplishments is his set of postulates for the natural numbers, which is the subject of this project.

Peano was a very prolific author. Throughout his professional career he published over 200 research papers and books. He was very actively involved in editorial work as well. In 1891 he founded the journal *Rivista di Matematica*, and was the main force behind cataloguing all the known theorems on the then mainstream subjects of mathematics—the project that became known as the *Formulario* project.

Peano created his own school of mathematics, the best known representatives of which are Cesare Burali-Forti (1861–1931), Mario Pieri (1860–1913), and Alessandro Padoa (1868–1937). He also actively corresponded with the leading mathematicians of his time, including Georg Cantor (1845–1918), Frege, Felix Klein (1849–1925), Russell and many others. His controversy with another famous professor at Turin, Vitto Volterra (1860-1940), is well-documented in [7, Chapter 8].

During his life Peano was a well recognized mathematician in Italy. In 1891 he was elected a member of the Academy of Sciences of Turin, and in 1905 he became a Corresponding Member of the *Accademia dei Lincei*—considered as one of the highest honors for an Italian scientist. In addition, he was first made a Knight of the Crown of Italy (1895), then an Officer of the Crown of Italy (1917), and finally *Commendatore* of the Crown of Italy (1921).

The international influence of Peano was perhaps best manifested at the 1900 International Congress of Philosophy in Paris, where Peano and his school dominated the discussion. The following is Russell's account of Peano at the congress and his effect on Russell, taken from [10, pp. 217–218]:

> The Congress was a turning point of my intellectual life, because I there met Peano... In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked.

Although the congress was the peak of Peano's prowess, it was also the starting point of his decline as a mathematician. His interests started to switch slowly towards artificial languages and later also towards mathematics education. At the time artificial languages were in fashion. Among the most popular ones were *Esperanto* and *Volapük*. Instead Peano proposed to use a simplified form of Latin, Latin stripped of grammar—*Latino sine flexione*. In 1908 he was unanimously elected director of the *Academia pro Interlingua*—an international academy of artificial languages—and remained its director until his death in 1932. Since his election his interests were mostly directed towards promoting the international auxiliary language movement and away from original mathematical research.

Peano died on 20 April, 1932. His last years were serene and satisfying. He was a well respected and revered figure in the mathematical community and international auxiliary language movement. Peano's gentle personality, his tolerance of human weakness, and his perennial optimism have been remembered by his family, friends, and disciples. For the rest of us, however, his name will always be associated with the set of postulates for the natural numbers, which we discuss now in detail.

3 Peano's Postulates

In 1889 Peano published a treatise *"Arithmetices principia, nova methodo exposita"* (The principles of arithmetic, presented by a new method), where his famous postulates for the natural numbers appeared for the first time. This project is based on its English translation which appeared in [1, pages 101–134].

The treatise consists of a preface and 10 sections. In the preface Peano explains his formalism and discusses the basic principles of logic and set theory that he uses throughout the treatise. Sections 1 through 7 develop the axiomatic theory of natural numbers and are our main concern. Sections 8 and 9 are dedicated to the theory of rational and real numbers, and finally section 10 presents some new theorems about sets of real numbers.

We begin by reading several passages from the preface to become familiar with Peano's notation. From the outset Peano states that his development is purely formal:

> I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs. The signs pertain either to logic or to arithmetic.

A little later he clearly expresses his belief in the axiomatic method:

> But in order to treat other theories, it is necessary to adopt new signs to indicate new entities. I believe, however, that with only these signs of logic the propositions of any science can be expressed, so long as the signs which represent the entities of the science are added.

A belief that the twentieth century proved to be true!

In his logical notation Peano uses "P" for *proposition*, "$\cap$" for *and*, "$\cup$" for *or*, "$-$" for *not*, "Λ" for *false* or *absurd*, and "$\supset$" for *one deduces*. For the sake of precision, Peano sometime uses "$\supset_{x,y,\ldots}$" instead of "$\supset$":

> If the propositions a, b contain the indeterminate quantities $x, y, \ldots$, that is, express conditions on these objects, then $a \supset_{x,y,\ldots} b$ means: whatever the $x, y, \ldots$, from proposition a one deduces b. If indeed there is no danger of ambiguity, instead of $a \supset_{x,y,\ldots} b$ we write only $\supset$.

In his set-theoretic notation Peano uses "K" for *class* (or *set* as we say today), and for the first time(!) introduces "$\in$" to denote *is a member of*. In addition, he uses the same symbols "$\supset$" and "Λ" that he used in his logical notation to denote *is a subset of* and *empty set*, respectively.[1] He also uses $[x \in]a$ for *those x such that a*. In his formalism Peano frequently uses dots instead of parentheses:

> Generally we write signs on the same line. So that it will be clear how they are to be joined, we use parentheses, as in algebra, or rather points . : $\therefore$ :: etc. So that a formula divided by points may be understood, first the signs which are not separated by points are taken together, then those separated by one point, then those by two points, etc. For example, let $a, b, c, \ldots$ be any signs. Then $ab \ . \ cd$ means $(ab)(cd)$; and $ab \ . \ cd \ : \ ef \ . \ gh \ \therefore \ k$ means $(((ab)(cd))((ef)(gh)))k$.

After explaining his notation in the preface, Peano begins section 1 by presenting his famous postulates for the natural numbers.[2]

> *Explanations*
>
> The sign N means *number* (*positive integer*); 1 means *unity*; $a + 1$ means the *successor of* a or a *plus* 1; and $=$ means *is equal to*...
>
> *Axioms*
>
> $1 \in \mathsf{N}$.
>
> $a \in \mathsf{N} \ . \ \supset \ . \ a + 1 \in \mathsf{N}$.
>
> $a, b \in \mathsf{N} \ . \ \supset : \ a = b \ . \ = \ . \ a + 1 = b + 1$.
>
> $a \in \mathsf{N} \ . \ \supset \ . \ a + 1 - = 1$.
>
> $k \in \mathsf{K} \ \therefore \ 1 \in k \ \therefore \ x \in \mathsf{N} \ . \ x \in k \ : \supset_x \ . \ x + 1 \in k \ :: \supset \ . \ \mathsf{N} \supset k$.

Exercise 1 Describe each postulate in your own words.

Note that although Peano did not include 0 in $\mathbb{N}$, today it is customary to do so.

[1] We note that the modern day notation for a subset relation is $\subset$, which is the opposite of Peano's notation. Also, today we denote the empty set by $\varnothing$.

[2] In fact, Peano had nine axioms, but four of the nine were expressing the usual properties of $=$ (such as $=$ is reflexive, symmetric, and transitive), which we decided to omit from the list below.

Exercise 2 Taking the above note into account, first describe the first-order language $\mathcal{L}$ for the five Peano postulates, and then express Peano's postulates in $\mathcal{L}$. How many axioms do you need to express Peano's postulates in $\mathcal{L}$? Explain why.

Exercise 3 Peano's fifth postulate is the celebrated principle of mathematical induction. Discuss the first-order axiomatization of the principle of mathematical induction. In particular, discuss whether the first-order axiomatization captures the full strength of the principle of mathematical induction.

Exercise 4 Do Peano's postulates have a model? Justify your answer.

Exercise 5 Are Peano's postulates independent? Justify your answer.

4 Addition

Peano's next task is to give a recursive definition of addition:

> *Definition*
>
> $a, b \in \mathsf{N} \ . \supset . \ a + (b + 1) = (a + b) + 1.$
>
> NOTE This definition should be read: if a and b are numbers, and $(a + b) + 1$ has meaning (that is, if $a + b$ is a number), but $a + (b + 1)$ has not yet been defined, then $a + (b + 1)$ indicates the number that follows $a + b$.

Exercise 6 Expand the first-order language $\mathcal{L}$ by a new binary function symbol $+$ for addition, and in the enriched first-order language $\mathcal{L}_+ = \mathcal{L} \cup \{+\}$, express the axioms defining addition.

At the end of section 1, Peano proves that $+$ is well defined (that is, $a, b \in \mathbb{N}$ imply $a + b \in \mathbb{N}$), that $+$ is associative (that is, $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{N}$), and that $+$ is commutative (that is, $a + b = b + a$ for all $a, b \in \mathbb{N}$).

Exercise 7 Let Φ denote the set of Peano postulates in $\mathcal{L}$ together with the two axioms in $\mathcal{L}_+$ defining $+$. Show that it is derivable from Φ that $+$ is associative.

Exercise 8 Show that it is derivable from Φ that $+$ is commutative. (Hint: First show that $(\forall x)(x + 0 = 0 + x)$ and $(\forall x)(x + 1 = 1 + x)$ are derivable from Φ, where 1 is an abbreviation of $s(0)$. To show that $(\forall x)(x + 1 = 1 + x)$ is derivable from Φ, first show that $(\forall x)(s(x) = x + s(0))$ is derivable from Φ.)

5 Subtraction and $<$

In section 2 Peano defines subtraction and $<$.

> *Explanations*
>
> The sign $-$ is read *minus*, $<$ is read *is less than*, and $>$ is read *is greater than*.
>
> *Definitions*
>
> $a, b \in \mathsf{N}\ .\ \supset:\ b - a = \mathsf{N}[x \in](x + a = b).$
>
> $a, b \in \mathsf{N}\ .\ \supset:\ a < b\ .\ =\ .\ b - a - = \Lambda.$
>
> $a, b \in \mathsf{N}\ .\ \supset:\ b > a\ .\ =\ .\ a < b.$

Exercise 9 Describe the above three definitions in your own words.

Note that since subtraction is only a partial function, in $\mathcal{L}_+$ it is easier to define $<$ directly from $+$.

Exercise 10 Define $<$ in $\mathcal{L}_+$.

In the remainder of section 2 Peano proves many basic properties of $<$. In particular, he shows:

$$a, b, a', b' \in \mathsf{N}\ .\ a < b\ .\ a' < b'\ \ :\supset\ \ .\ a + a' < b + b'.$$

And proves that $<$ is a strict linear order.

Exercise 11 Express the above theorem in $\mathcal{L}_+$ and show that it is derivable from Φ. Also show that it is derivable from Φ that $<$ is a strict linear order.

6 Multiplication

Peano gives a recursive definition of multiplication in section 4 of his treatise.

> *Definitions*
>
> $a \in \mathsf{N}\ .\ \supset\ .\ a \times 1 = a.$
>
> $a, b \in \mathsf{N}\ .\ \supset\ .\ a \times (b + 1) = a \times b + a.$
>
> $ab = a \times b.$
>
> $ab + c = (ab) + c.$

Exercise 12 Expand further the first-order language $\mathcal{L}_+$ by a new binary function symbol $\times$ for multiplication, and in the enriched first-order language $\mathcal{L}_{+,\times}$ express the axioms defining multiplication.

Let PA denote the set Φ together with the two axioms in $\mathcal{L}_{+,\times}$ defining multiplication.

Definition 13 The first-order theory PA is called *Peano Arithmetic.*

Note that PA is strictly stronger than the first-order theory of arithmetic developed without addition and/or multiplication.

Peano shows that multiplication is well defined and establishes many basic properties of multiplication, such as commutativity, associativity, distributivity over +, and others. In particular, he proves the following theorems:

$$a, b \in \mathsf{N} \ . \ \supset \ . \ ab = ba.$$

$$a, b, c \in \mathsf{N} \ . \ \supset \ . \ a(b + c) = ab + ac.$$

$$a, b, a', b' \in \mathsf{N} \ . \ a < a' \ . \ b < b' \ : \supset : \ ab < a'b'.$$

$$a, b, c \in \mathsf{N} \ . \supset \ . \ a(bc) = (ab)c.$$

Exercise 14 First describe the above four theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA. (Hint: In showing that it is provable in PA that multiplication is commutative, associative, and distributes over addition, start by showing that $\mathsf{PA} \vdash (\forall x, y, z)(x(yz) = (xy)z)$ and $\mathsf{PA} \vdash (\forall x, y, z)(x(y+z) = xy+xz)$. Then show that $\mathsf{PA} \vdash (\forall x)(0x = 0)$ and $\mathsf{PA} \vdash (\forall x)(x1 = 1x)$. Finally, prove that $\mathsf{PA} \vdash (\forall x, y)(xy = yx)$.)

7 Division

Peano defines division in section 6 of his treatise.

Explanations

The sign / is read *divided by*, D is read *divides*, or *is a divisor of*, $\mathsf{Ɑ}$ is read *is a multiple of*, Np is read *prime number*, and π is read *is prime with*.

Definitions

$$a, b \in \mathsf{N} \ . \ \supset \ . \ b/a = \mathsf{N}[x \in](xa = b).$$

$$a, b \in \mathsf{N} \ . \ \supset : \ a\mathsf{D}b \ . \ = \ . \ b/a - = \Lambda.$$

$$a, b \in \mathsf{N} \ . \ \supset : \ b\mathsf{Ɑ}a \ . \ = \ . \ a\mathsf{D}b.$$

$$\mathsf{Np} = \mathsf{N}[x \in](\ni \mathsf{D}\ x \ . \ \ \ni > 1 \ . \ \ \ni < x \ \ : = \Lambda).$$

$$a, b \in \mathsf{N} \ . \ \supset :: \ a\pi b \ \therefore = \therefore \ni \mathsf{D}\ a \ . \ \ \ni \mathsf{D}\ b \ . \ \ \ni > 1 \ \ : = \Lambda.$$

Exercise 15 Describe the above definitions in your own words.

Note that division is only a partial function.

Exercise 16 Express D, Ɑ, Np, and π in the first-order language of PA. (Hint: Note that Peano's definition of prime numbers does *not* rule out that 1 is a prime number. Nevertheless, in expressing Np, you should follow the standard assumption that 1 is *not* a prime number.)

Peano concludes section 6 by establishing many basic properties of division. In particular, he proves the following theorems:

$$a \in \mathsf{N} \;.\; \supset \;.\; 1\mathsf{D}a.$$

$$a \in \mathsf{N} \;.\; \supset \;.\; a\mathsf{D}a.$$

$$a, b, c \in \mathsf{N} \;.\; a\mathsf{D}b \;.\; b\mathsf{D}c \;:\supset\;.\; a\mathsf{D}c.$$

$$a, b, c, m, n \in \mathsf{N} \;.\; c\mathsf{D}a \;.\; c\mathsf{D}b \;:\supset\;.\; c\mathsf{D}ma + nb.$$

Exercise 17 First describe the above theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA.

In a short section 7 Peano gives an extensive list of well-known theorems about integers that can be derived from his postulates, including several theorems from Euclid's *Elements* and Fermat's Little Theorem. The following theorems are part of the list:

$$x \in \mathsf{N} \;.\; \supset \;.\; x(x+1)\text{ᗡ}2.$$

$$x \in \mathsf{N} \;.\; \supset \;.\; x(x+1)(x+2)\text{ᗡ}6.$$

$$a, b \in \mathsf{N} \;.\; a \in \mathsf{Np} \;.\; a \;\; -\mathsf{D}b \;:\supset:\; a\pi b.$$

$$a, b, c \in \mathsf{N} \;.\; a\pi b \;.\; c\mathsf{D}a \;:\supset\;.\; c\pi b.$$

Exercise 18 First describe the above theorems in your own words, then express them in the first-order language of PA, and finally derive them in PA.

8 Peano Arithmetic

In this final section of the project we will summarize the postulates Peano developed for axiomatic treatment of arithmetic, and will also give a modern analysis of his postulates.

As we saw, in order to express Peano's five postulates, our first-order language needs to have the successor function s and the constant 0. Also, the formal treatment of addition and multiplication requires that we expand our first-order language with two binary functions $+$ and $\times$, and add to Peano's five postulates four more postulates, two of which govern the behavior of addition and the other two that of multiplication. The resulted system is known as Peano Arithmetic and is denoted PA. In PA we can develop arithmetic on a purely formal basis. In particular, all the basic properties of addition and multiplication, such as associativity, commutativity, and distributivity of multiplication over addition become theorems of PA. We can also define subtraction, division, and $<$ in PA and prove their basic properties.

The key postulate that allows us to perform all these proofs successfully is Peano's fifth postulate which expresses the principle of mathematical induction. Unfortunately, first-order languages cannot express the full strength of the principle of mathematica induction. The reason being that we cannot quantify over formulas of a first-order language. Instead, the first-order version of Peano's fifth postulate is an axiom-schema of mathematical induction. This leads to some paradoxical results, such as Skolem's Theorem, which we will discuss now.

Let $\mathbb{N}$ be the set of natural numbers. Then $(\mathbb{N}, s, 0)$ is a model of Peano's five postulates, where $s : \mathbb{N} \to \mathbb{N}$ is the successor function. Let $(A, s^A, 0^A)$ be another model of Peano's five postulates. The question that we ask is whether $(\mathbb{N}, s, 0)$ is isomorphic to $(A, s^A, 0^A)$? In other words, does there exist a bijection $f : \mathbb{N} \to A$ such that $f(0) = 0^A$ and $f(s(n)) = s^A(f(n))$ for each $n \in \mathbb{N}$?

The answer is yes provided that we can use the full strength of the principle of mathematical induction. In fact, we can define $f : \mathbb{N} \to A$ by induction as follows: let $f(0) = 0^A$, and if $f(n)$ is already defined for $n \in \mathbb{N}$, then let $f(s(n)) = s^A(f(n))$.

Exercise 19 Use the principle of mathematical induction for $(A, s^A, 0^A)$ to prove that f is onto.

Exercise 20 Prove that for all $m, n \in \mathbb{N}$, if $m \neq n$, then $f(m) \neq f(n)$. (Hint: Induction on n.)

Exercise 21 Conclude that f is an isomorphism.

What you have just established is known as *Dedekind's Theorem.* Note that Dedekind's Theorem requires the full strength of the principle of mathematical induction, which cannot be formalized in a first-order language. This has some unusual consequences. The Norwegian mathematician Thoralf Skolem (1887-1963) showed that the first-order theory of arithmetic has models which are not isomorphic to $\mathbb{N}$. In fact, Skolem's Theorem states that the first-order theory of arithmetic has even countable models which are not isomorphic to $\mathbb{N}$.

In order to see why, we will require the compactness and Löwenheim-Skolem theorems. The compactness theorem states that if each finite subset of a set Σ of first-order formulas is satisfiable, then Σ is satisfiable. The Löwenheim-Skolem theorem states that if Σ has an infinite model, then Σ has models of any infinite cardinality. The compactness and Löwenheim-Skolem theorems form the core of first-order logic, and can be found in every textbook on first-order logic; see, e.g., [4]. They are also discussed in the following historical project [2].

Let $\mathrm{Th}(\mathbb{N}, s, 0)$ be the first-order theory of $(\mathbb{N}, s, 0)$; that is, the set of first-order sentences in $\mathcal{L}$ satisfiable in $(\mathbb{N}, s, 0)$. Let $\Sigma = \mathrm{Th}(\mathbb{N}, s, 0) \cup \{\neg(x = n) : n \in \mathbb{N}\}$.

Exercise 22 Prove that Σ has a model. (Hint: Show that each finite subset of Σ can be satisfied in $(\mathbb{N}, s, 0)$. Then use the compactness theorem.)

Exercise 23 Prove that each model of Σ is infinite.

Exercise 24 Prove that Σ has a countable model. (Hint: Use the Löwenheim-Skolem theorem.

Exercise 25 Prove that the countable model of Σ you have constructed is not isomorphic to $(\mathbb{N}, s, 0)$. Conclude that the first-order theory of arithmetic has a countable model not isomorphic to $(\mathbb{N}, s, 0)$. Such models are called *nonstandard models.*

What you have just established is known as *Skolem's Theorem.* Another formulation of Skolem's Theorem is that the first-order theory of arithmetic has countable nonstandard models.

9 Notes to the Instructor

The project is designed for an upper level undergraduate course in mathematical logic. Although the project is relatively short, some of the exercises are not as easy as they appear. Instructors may wish to spend some class time on discussing how formal proofs should be conducted, and also provide some additional assistance with completing some of the exercises. The last part of the project assumes that students are familiar with the compactness and Löwenheim-Skolem theorems. I usually assign the project at the end of the semester, after covering the completeness, Löwenheim-Skolem, and compactness theorems. The entire project usually takes about three to four weeks. The notation Peano uses in his treatise is slightly outdated from today's point of view. Instructors may wish to spend some time comparing Peano's notation to the contemporary one.

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