Historical Projects in Discrete Mathematics and Computer Science

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1 Background

In the United States a course in discrete mathematics is a relatively recent addition, within the last 30 or 40 years, to the undergraduate mathematics curriculum. The course serves to instruct students in precise logical and algorithmic thought, needed for further study of modern mathematics or computer science. The course is also required of many secondary education majors, who will teach mathematics in middle or high schools. The roots of discrete mathematics, however, are as old as mathematics itself, with the notion of counting a discrete operation, usually cited as the first mathematical development in ancient cultures [7]. By contrast, a course in finite mathematics is frequently presented as a fast-paced news reel of facts and formulae, often memorized by the students, with the text offering only passing mention of the motivating problems and original work that eventually found resolution in the modern concepts of induction, recursion and algorithm. This paper focuses on the pedagogy of historical projects that offer excerpts from primary historical sources, place the material in context, and provide direction to the subject matter.

2 Classroom Projects

Each historical project is centered around a publication of mathematical significance, such as Blaise Pascal's "Treatise on the Arithmetical Triangle" [9, vol. 30] from the 1650s or Alan Turing's 1936 paper "On Computable Numbers with an Application to the Entscheidungsproblem" [12]. The projects are designed to introduce or provide supplementary material for topics in the curriculum, such as induction in a discrete mathematics course, or compilers and computability for a computer science course. Each project provides a discussion of the historical exigency of the piece, a few biographical comments about the author, excerpts from the original work, and a sequence of questions to help the student appreciate the source. The main pedagogical idea is to teach and learn certain course topics from the primary historical source, thus recovering motivation for studying the material.

To lend the reader a sense of mathematical scope, our team has written about a dozen historical projects, listed below together with the primary historical author whose work is highlighted in the module. The projects are slated to appear in print [1], and are presently available through the web resource [2].

1. "Are All Infinities Created Equal?" (Georg Cantor, 1845–1918, [4])
2. "Turing Machines, Induction and Recursion" (Alan Turing, 1912–1954, [12])
3. "Turing Machines and Binary Addition" (Alan Turing, 1912–1954, [12])

4. "Binary Arithmetic: From Leibniz to von Neumann" (Gottfried Leibniz, 1646–1716, [6])
5. "Arithmetic Backwards from von Neumann to the Chinese Abacus" (Claude Shannon, 1916–2001, [10])
6. "Treatise on the Arithmetical Triangle" (Blaise Pascal, 1623–1662, [9])
7. "Counting Triangulations of a Polygon" (Gabriel Lamé, 1795–1870, [8])
8. "Two-Way Deterministic Finite Automata" (John Shepherdson [11])
9. "Church's Thesis" (Alonzo Church, 1903–1995, [5])
10. "Euler Circuits and the Königsberg Bridge Problem" (Leonhard Euler, 1707–1783, [3])
11. "Topological Connections from Graph Theory" (Oswald Veblen, 1880–1960, [3])
12. "Hamiltonian Circuits and Icosian Game" (William Hamilton, 1805–1865, [3])

This paper outlines the content of two of the projects "Treatise on the Arithmetical Triangle," and "Counting Triangulations of a Polygon," designed to teach students induction and topics in combinatorics. Often in high schools and introductory discrete mathematics courses at universities, induction is taught as a formal procedure, with no regard to its historical development or discovery. The first project begins with excerpts from Blaise Pascal's *Traité du Triangle Arithmetique* (Treatise on the Arithmetical Triangle), in which he arranges the figurate numbers, i.e., the linear numbers, the triangular numbers, the pyramidal numbers, etc., in adjacent columns of one table, resulting in "Pascal's Triangle." This simple organizing devise allows Pascal to identify many patterns in his table, which he wishes to show continue no matter how far the table is extended. Showing unusual rigor for his day, Pascal develops a system of reasoning, stated verbally, that is easily recognized as the modern principle of mathematical induction. In the 12th consequence of his treatise, Pascal wishes to prove that the ratio of two consecutive entries in a given base of the triangle has a particular expression. To prove this, he states that the ratio is found in the beginning base, and that if the ratio holds in some base, then it will necessarily be found in the following base. Having students grapple with the logic of this statement and test their conjectures with concrete entries from the triangle is a wonderful learning exercise. Moreover, the particular ratio that Pascal identifies is key to the development of the modern formula for binomial coefficients, $\binom{r}{s}$. Sample student exercises are numbered 1–10 in the excerpt from the project below.

Treatise on the Arithmetical Triangle

As a point of departure, we study the figurate numbers to glean an understanding of Pascal's triangle. These numbers count the number of dots in certain regularly shaped geometric figures. In particular $Z(n,\ k)$ represents the nth figurate number in dimension k. For $k = 2$, the triangular numbers are:

$Z(1,\ 2) = 1 \qquad Z(2,\ 2) = 3 \qquad Z(3,\ 2) = 6 \qquad Z(4,\ 2) = 10.$

In general, the nth triangle is formed by placing the $(n-1)$st triangle on a line segment with n dots.

The pyramidal numbers, when $k = 3$, count the number of dots in certain regularly shaped pyramids. We have

$$Z(1,\ 3) = 1, \quad Z(2,\ 3) = 4, \quad Z(3,\ 3) = 10, \quad Z(4,\ 3) = 20.$$

In general, the nth pyramid is formed by placing the $(n-1)$st pyramid on the nth triangle.

1. Compute $Z(5,\ 2)$ and $Z(5,\ 3)$. Be sure to justify your answer. Sketch the fifth triangle and the fifth pyramid in these sequences as well.

The nth figurate number in dimension k is constructed by placing the $(n-1)$st figure in dimension k on the nth figure in dimension one less, $k-1$, resulting in the recursion relation, expressed in modern notation as:

$$Z(n,\ k) = Z(n,\ k-1) + Z(n-1,\ k), \quad n \geq 1,\ k \geq 0, \text{ and } (n,\, k) \neq (1,\, 0),$$

subject to certain conditions to initialize these numbers. Set $Z(1,\ 0) = 1$, since the first number in dimension zero corresponds to one dot. Then set $Z(0,\ k) = 0$ for $k \geq 1$ and $Z(n,\ -1) = 0$ for $n \geq 2$.

2. Explain how $Z(4,\ 5)$ can be computed from this recursion relation.

Pascal's genius was to organize the figurate numbers into one table, beginning with the zero-dimensional numbers ($k = 0$) in the first column, the linear numbers ($k = 1$) in the second column, the triangular numbers ($k = 2$) in the third column, etc.

Let's read from Blaise Pascal's

TREATISE ON THE ARITHMETICAL TRIANGLE

Definitions

I call *arithmetical triangle* a figure constructed as follows:

From any point, G, I draw two lines perpendicular to each other, GV, Gζ in each of which I take as many equal and contiguous parts as I please, beginning with G, which I number 1, 2, 3, 4, etc., and these numbers are the *exponents* of the sections of the lines.

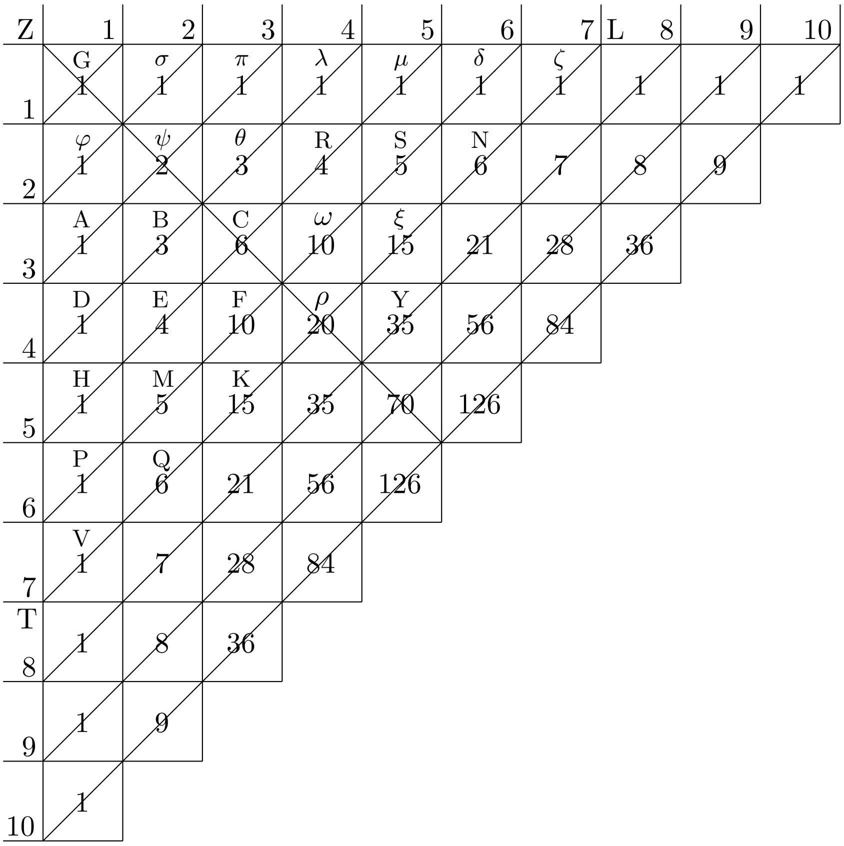
Next I connect the points of the first section in each of the two lines by another line, which is the base of the resulting triangle.

In the same way I connect the two points of the second section by another line, making a second triangle of which it is the base.

And in this way connecting all the points of section with the same exponent, I construct as many triangles and bases as there are exponents.

Through each of the points of section and parallel to the sides I draw lines whose intersections make little squares which I call *cells*.

Cells between two parallels drawn from left to right are called *cells of the same parallel row*, as, for example, cells G, σ, π, etc., or φ, ψ, θ, etc.



3. How are the cells G, σ, π, φ, ψ, and θ labeled in the $Z(n,\ k)$ notation? Find values of n and k that correspond to each of these letters.

Those between two lines are drawn from top to bottom are called *cells of the same perpendicular row* [column], as, for example, cells G, φ, A, D, etc., or σ, ψ, B, etc.

4. How are the cells G, φ, A, D, σ, ψ, and B labeled in the $Z(n,\ k)$ notation? Find values of n and k that correspond to each of these letters.

Those cut diagonally by the same base are called *cells of the same base*, as, for example, D, B, θ, λ, or A, ψ, π.

5. How are cells in the same base related to each other in terms of the $Z(n,\ k)$'s? Verify your claim with the cells A, ψ, π, and the cells D, B, θ, λ.

. . . . Now the numbers assigned to each cell are found by the following method:

The number of the first cell, which is at the right angle, is arbitrary; but that number having been assigned, all the rest are determined, and for this reason it is called the *generator* of the triangle. Each of the others is specified by a single rule as follows:

The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately preceding. Thus cell F, that is, the number of cell F, equals the sum of cell C and cell E, and similarly with the rest.

6. Write $F = C+E$ in the $Z(n,\ k)$ notation. Also express "The number of each cell is equal to the sum of the numbers of the perpendicular and parallel cells immediately

preceding" in terms of the $Z(n,\ k)$'s. This will be called the *construction principle* for Pascal's triangle. How does the construction principle compare to the recursive definition of the $Z(n,\ k)$'s ?

Whence several consequences are drawn. The most important follow, wherein I consider triangles generated by unity, but what is said of them will hold for all others.

First Consequence

In every arithmetical triangle all the cells of the first parallel row and of the first perpendicular row [column] are the same as the generating cell. ...

Second Consequence

In every arithmetical triangle each cell is equal to the sum of all the cells of the preceding parallel row from its own perpendicular row to the first, inclusive.

Let any cell, ω, be taken. I say that it is equal to $R + \theta + \psi + \varphi$, which are the cells of the next higher parallel row from the perpendicular row of ω to the first perpendicular row.

This is evident if we simply consider a cell as the sum of its component cells.

For ω equals $R + \underbrace{C}_{\theta\ +\ \underbrace{B}_{\psi\ +\ \underbrace{A}_{\varphi,}}}$

for A and φ are equal to each other by the preceding consequence.

Therefore $\omega = R + \theta + \psi + \varphi$.

7. Express the statement of the second consequence in $Z(n,\ k)$ notation. Rewrite Pascal's computation of ω entirely in terms of the $Z(n,\ k)$'s, indicating why $\omega = R + C$, $C = \theta + B$, $B = \psi + A$. Express all of these equations using the $Z(n,\ k)$'s. Write a proof of Pascal's second consequence for an arbitrary cell $Z(n,\ k)$, based on his computation of ω. Comment on whether you feel that Pascal has proven the second consequence based on an argument for one entry in the triangle.

The Twelfth Consequence and Mathematical Induction

Certainly the numbers in any row of Pascal's triangle follow a predictable pattern, since the difference between any two consecutive entries in a given row, i.e., $Z(n,\ k+1) - Z(n,\ k)$, is simply the entry in the cell above the larger of these numbers, i.e.,

$$Z(n,\ k+1) - Z(n,\ k) = Z(n-1,\ k+1).$$

Do you know why this equation holds? Thus, the first differences of the entries in any row are the entries in the above row of the triangle. Also, the entries in any column of Pascal's triangle follow a very similar pattern for their first differences. Are there any patterns across a diagonal base? Experiment a bit by forming the first differences along a diagonal. Do you see any patterns?

Form the quotient of consecutive entries along a diagonal. Do you see any patterns here? The next consequence is the most important and famous in the whole treatise. Pascal derives a formula for the ratio of consecutive numbers in a base of the triangle. From this he will obtain an elegant and efficient formula for all the numbers in the triangle.

Twelfth Consequence

In every arithmetical triangle, of two contiguous cells in the same base the upper is to the lower as the number of cells from the upper to the top of the base is to the number of cells from the lower to the bottom of the base, inclusive.

Let any two contiguous cells of the same base, E, C, be taken. I say that

E	:	C	::	2	:	3
the lower		the upper		because there are two cells from E to the bottom, namely E, H,		because there are three cells from C to the top, namely C, R, μ.

8. Express the twelfth consequence concerning the ratio of two consecutive entries in the same base using $Z(n,\ k)$ notation. Be mindful that k denotes the dimension of a figurate number, and is not a column number. Hint: Study the values of $Z(n,\ k)/Z(n+1,\ k-1)$ along a base.

Although this proposition has an infinity of cases, I shall demonstrate it very briefly by supposing two lemmas:

The first, which is self-evident, that this proportion is found in the second base, for it is perfectly obvious that $\varphi : \sigma :: 1 : 1$;

The second, that if this proportion is found in any base, it will necessarily be found in the following base.

Whence it is apparent that it is necessarily in all the bases. For it is in the second base by the first lemma; therefore by the second lemma it is in the third base, therefore in the fourth, and to infinity.

It is only necessary therefore to demonstrate the second lemma as follows: If this proportion is found in any base, as, for example, in the fourth, $D\lambda$, that is, if $D : B :: 1 : 3$, and $B : \theta :: 2 : 2$, and $\theta : \lambda :: 3 : 1$, etc., I say the same proportion will be found in the following base, $H\mu$, and that, for example, $E : C :: 2 : 3$.

For $D : B :: 1 : 3$, by hypothesis.

Therefore $\underbrace{D+B}_{E} : B :: \underbrace{1+3}_{4} : 3$

$$E : B :: 4 : 3$$

Similarly $B : \theta :: 2 : 2$, by hypothesis

Therefore $\underbrace{B+\theta}_{C} : B :: \underbrace{2+2}_{4} : 2$

$$C : B :: 4 : 2$$

But $B : E :: 3 : 4$

Therefore, by compounding the ratios, $C : E :: 3 : 2$. Q.E.D.

The proof is the same for all other bases, since it requires only that the proportion be found in the preceding base,and that each cell be equal to the cell before it together with the cell above it, which is everywhere the case.

9. Prove Pascal's claim that "if this proportion is found in any base, [then] it will necessarily be found in the following base." Since this is an "if-then" statement, begin by supposing that the proportion holds in the Lth base ($L \geq 2$). How are the two entries of $Z(n,\ k)$ related for cells in the Lth base? Then find a step-by-step verification that the same ratio holds in the $(L+1)$st base. Use the hypothesis of the if statement, the construction principle for Pascal's triangle, and his example above to prove the conclusion.
10. The gist of Pascal's argument is that the ratio holds in the first base where a ratio can be computed, i.e., the second base, and that if the ratio holds in some base, then it necessarily holds in the next base. How can you conclude from this that the ratio holds in all bases of the triangle, even in those not pictured in Pascal's diagram of the first ten bases of the triangle?

Counting Triangulations of a Polygon

As an exploration in combinatorics, this second project develops a closed formula for the number of triangulations of a convex polygon based on Gabriel Lamé's solution of the recursion relation for these numbers established earlier by Euler and others. Unlike the treatment in modern textbooks involving generating functions, Lamé's solution exploits the geometry of a convex n-gon to arrive at two points of view, one based on triangulations containing a fixed triangle, and one based on triangulations containing a fixed diagonal. The resulting two algebraic equations contain similar terms, and when one is substituted for the other, a formula emerges that easily reduces to the modern equation for the Catalan numbers. When used as a student module, the project allows the student to gain insight into the discovery process, and provides a context for the development of these numbers. What follows are a few excerpts from Lamé's paper [8] as well as a few questions from the student module.

Excerpt from a letter of Monsieur Lamé to Monsieur Liouville on the question: *Given a convex polygon, in how many ways can one partition it into triangles by means of diagonals?*

"The formula that you communicated to me yesterday is easily deduced from the comparison of two methods leading to the same goal.

"Indeed, with the help of two different methods, one can evaluate the number of decompositions of a polygon into triangles: by consideration of the sides, or of the vertices.

I.

"Let $ABCDEF\ldots$ be a convex polygon of $n+1$ sides, and denote by the symbol P_k the total number of decompositions of a polygon of k sides into triangles. An arbitrary side AB of $ABCDEF\ldots$ serves as the base of a triangle, in each of the P_{n+1} decompositions of the polygon, and the triangle will have its vertex at C, or D, or $F\ldots$; to the triangle CBA there will correspond P_n different decompositions; to DBA another group of decompositions, represented by the product P_3P_{n-1}; to EBA the group P_4P_{n-2}; to FBA, P_5P_{n-3}; and so forth, until the triangle ZAB, which will belong to a final group P_n. Now, all these groups are completely distinct: their sum therefore gives P_{n+1}. Thus one has

$$(1) \qquad P_{n+1} = $$
$$P_n + P_3P_{n-1} + P_4P_{n-2} + P_5P_{n-3} + \cdots + P_{n-3}P_5 + P_{n-2}P_4 + P_{n-1}P_3 + P_n.$$

2.1. Explain why the triangulations belonging to the groups

$$P_n,\ P_3P_{n-1},\ P_4P_{n-2},\ \ldots,\ P_{n-1}P_3,\ P_n$$

are distinct.

2.2. Does every triangulation of a convex polygon with $n+1$ sides occur in one of the groups represented by

$$P_n,\ P_3P_{n-1},\ P_4P_{n-2},\ \ldots,\ P_{n-1}P_3,\ P_n\ ?$$

Why or why not?

II.

"Let $abcde\ldots$ be a polygon of n sides. To each of the $n-3$ diagonals, which end at one of the vertices a, there will correspond a group of decompositions, for which this diagonal will serve as the side of two adjacent triangles: to the first diagonal ac corresponds the group P_3P_{n-1}; to the second ad corresponds P_4P_{n-2}; to the third ae, P_5P_{n-3}, and so forth until the last ax, which will occur in the group P_3P_{n-1}. These groups are not totally different, because it is easy to see that some of the partial decompositions, belonging to one of them, is also found in the preceding ones. Moreover they do not include the partial decompositions of P_n in which none of the diagonals ending in a occurs.

Lamé's use of diagonals leads to an enumeration of triangulations which is neither one-to-one nor inclusive of all triangulations. His genius, however, was to slightly alter this strategy to first include all triangulations, and then to count how many times a generic triangulation occurs. Combined with the results of §I, this results in a streamlined computation for P_n.

"But if one does the same for each of the other vertices of the polygon, and combines all the sums of the groups of these vertices, by their total sum

$$n(P_3P_{n-1}+P_4P_{n-2}+\cdots+P_{n-2}P_4+P_{n-1}P_3)$$

one will be certain to include all the partial decompositions of P_n; each of these is itself repeated therein a certain number of times. ...

Lamé continues and claims that each triangulation is counted $2n-6$ times in

$$n(P_3P_{n-1}+P_4P_{n-2}+\cdots+P_{n-2}P_4+P_{n-1}P_3),$$

from which he concludes $P_{n+1}=\frac{4n-6}{n}P_n$. A student exercise is to use the above equation to find a formula for P_{n+1} in terms of binomial coefficients.

3 Implementation

For use in the classroom, allow several weeks per project with one or two projects per course. Each project should count for a significant portion of the course grade (about 20%) and may take the place of an in-class examination. Begin early in the course with a discussion of the relevance of the historical piece, its relation to the course curriculum, and how modern textbook techniques owe their development to problems often posed centuries earlier. While a project is assigned, several class activities are possible. Students could be encouraged to work on the project in class, either individually or in small groups, as the instructor monitors their progress and explores meaning in language from time past. A comparison with modern techniques could begin as soon as the students have read the related historical passages. For example, after reading Pascal's verbal description of what today is recognized as induction, the instructor could lead a discussion comparing this to the axiomatic formulation of induction found in the textbook. Finally, the historical source can be used to provide discovery exercises for related course material. In his 1703 publication "An Explanation of Binary Arithmetic" [6], Gottfried Leibniz introduces the binary system of numeration, states its advantages in terms of efficiency of calculation, and claims that this system allows for the discovery of other properties of numbers, such as patterns in the base two expansion of the perfect squares. An engaging in-class exercise is to examine patterns in a table of perfect squares (base two) and conjecture corresponding divisibility properties of the integers. The pattern of zeroes in the binary equivalent of n^2 leads to the conjecture that $8|(n^2-1)$, n odd, where the vertical bar denotes "divides." Construct the table! Time spent working on the project is time for explanation, exploration, and discovery, for both the instructor and the student. Instructors are encouraged to adapt each project to their particular course. Add or rephrase some questions, or delete others to reflect what is actually being covered. Be familiar with all details of a project before assignment. The source file for each project together with its bibliographic references can be downloaded and edited from the web resource [2].

4 Conclusion

In an initial pilot study of students learning discrete mathematics from primary historical sources, there were 229 cases where students earned course grades above the mean in subsequent courses, compared with 123 cases where students earned course grades below the mean in follow-on courses. The probability that this would occur under the assumption that the historical projects had no positive effect is less than .000001 using a simple binomial sign test. Of course there may be other factors at play, e.g., differing entering preparation for different groups of students in different courses and semesters. Furthermore, after completion of a course using historical projects, students write the following about the benefits of history: "See how the concepts developed and understand the process." "Learn the roots of what you've come to believe in." "Appropriate question building." "Helps with English-math conversion." "It leads me to my own discoveries."

ACKNOWLEDGMENT

The development of curricular materials for discrete mathematics and computer science has been partially supported by the National Science Foundation's Course, Curriculum and Laboratory Improvement Program under grant DUE-0231113, for which the authors are most appreciative. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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