Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization

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Notes to the instructor

This project is designed for an introductory or intermediate course in discrete or finite mathematics that considers boolean algebra from either a mathematical or computer science perspective. Some or all of the project could also be used in a course on abstract algebra or model theory. The project does assume some (minimal) familiarity with the set operations of union and intersection and their representation by Venn diagrams. This pre-requisite material may be gained by completing the companion (Boole) project described below, through reading a standard textbook treatment of elementary set operations, or via a short class discussion/lecture.

Although primarily based on an early paper in the study of axiomatizations of boolean algebras, the introductory section of this project provides a concise overview of George Boole's original work on 'the logic of classes' in order to provide students with a connection to a concrete example of a boolean algebra on which they can draw. In section 2, the goal of formal axiomatics is introduced through select readings from Huntington's 1904 paper *Sets of Independent Postulates for the Algebra of Logic*. Section 2 also introduces the now standard axioms for the boolean algebra structure and illustrates how to use these postulates to prove boolean algebra basic properties. Specific project questions also provide students with practice in using symbolic notation, and encourage them to analyze the logical structure of quantified statements. In Section 3, the use of models is illustrated in Huntington's use of the two-valued Boolean algebra on $K = \{0, 1\}$ — first studied by George Boole in his work on the logic of classes — to establish the *independence* and *consistency* of one of his postulate sets. The final section of this project discusses modern (undergradute) notation and axioms for boolean algebras, and provides several practice exercises to reinforce the ideas developed in the earlier sections.

Two other projects on boolean algebra are available as companions to this project, either or both of which could also be used independently of this project. The first companion project "Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce," is suitable as a preliminary to the Huntington project. Without explicitly introducing modern notation for operations on sets (until the concluding section), that project develops a modern understanding of these operations and their basic properties within the context of early efforts to develop a symbolic algebra for logic. By steadily increasing the level of abstraction, that project also lays the ground work for a more abstract discussion of a boolean algebra as a discrete structure. Other project questions prompt students to explore a variety of other mathematical themes, including the notion of an inverse operation, issues related to mathematical notation, and standards of rigor and proof.

The second companion project "Applying Boolean Algebra to Circuit Design: Claude Shannon" is suitable as either as a preliminary to or as a follow-up to the Huntington project on axioma-

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tization. Based on Shannon's groundbreaking paper *A Symbolic Analysis of Relay and Switching Circuits*, that project begins with a concise overview of the two major historical antecedents to Shannon's work: Boole's original work in logic and Huntington's work on axiomatization. The project then develops standard properties of a boolean algebra within the concrete context of circuits, and provides students with practice in using these to simplify boolean expressions The two-valued boolean algebra on $K = \{0, 1\}$ again plays a central role in this work. The project closes with an exploration the concept of a 'disjunctive normal form' for boolean expressions, again within the context of circuits.

Implementation with students of any of these projects may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. For the Huntington project, the instructor is encouraged to provide students with copies of Huntington's axioms and the definitions of his various models for consistency and independence on a separate sheet; these are included in Appendices B and C of this project.