Abstract awakenings in algebra: Early group theory in the works of Lagrange, Cauchy, and Cayley

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Notes to the instructor

This project of special length is designed to develop a significant portion of the core topics from elementary group theory from the standard curriculum of a one semester junior-level course in Abstract Algebra.[1] Topics developed in the project include roots of unity, permutations, definition and elementary properties of group (including results related to the order of group elements), abelian groups,cyclic groups, symmetric groups, alternating groups, Cayley tables, Lagrange's Theorem, group isomorphisms, classification of groups of small order, and direct products. The concept of cosets are also introduced in the main body of the project, and further developed in an appendix which also states the definitions of normal subgroup and factor group.

The project has been successfully site-tested as a replacement for a textbook during the elementary group theory of an Abstract Algebra course at several institutions. In each case, completion of the project required approximately half the term, with implementation taking place in essentially two ways:

- Develop most of elementary group theory (up to the introduction of factor groups) via the project during the first portion (approximately 60%) of the term, with other topics (e.g., factor groups, elementary ring theory) taught via a standard textbook during latter part of course.
- Begin the course with a study of ring theory taught via a standard textbook, followed by elementary group theory taught via the project during latter portion (approximately 40%) of course.

Note that completion of the project requires less time with students who are already familiar with ring theory. However, the project works equally well with (and was originally intended for) students who have no prior experience with either groups or rings.

Implementation of the project may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project. The manner in which student exercises are incorporated as an integral part of the text marks a distinctive pedagogical feature of the project. In contrast to the standard practice of placing exercises only at the end of each section or chapter, exercises which require students to actively engage with the mathematics as they read and work through each excerpt are interspersed throughout the project. Other exercises are intended to extend the original source excerpts by prompting students to fill in missing proof details and to reflect on the standards of rigor and style of proof illustrated by the excerpt.

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[1]Although the project assumes a minimum pre-requisite of linear algebra, the project has been used successfully as an one-year independent study with a student who had completed only Calculus I.

Through such exercises, students' ability to construct proofs in keeping with present-day standards is progressively extended throughout the project. For some topics, exercises are also provided at the end of the section in which they are introduced to assist students in further consolidating their understanding of the material.

The major feature which distinguishes the project from currently available textbooks is its extensive use of excerpts from original source material close to or representing the discovery of key algebraic concepts as a means to develop the material in question. Through guided reading of the excerpts and exercises based upon them, students will be prompted to explore these concepts and develop their own understanding of the underlying theory. Each excerpt is introduced by brief historical comments and biographical information about its author to set it in context and offer students a view of the humanistic aspect of mathematics. The primary merit of using original source excerpts to frame the material, however, lies in the pedagogical value of the original source excerpts with respect to providing context, motivation, and direction for the mathematical ideas to be developed. Accordingly, both the selection of primary source material and the design of the text narrative and supporting exercises focuses on providing an authentic point of departure for students' work and a direction for their efforts as they develop an understanding of the present-day paradigm of abstract algebra.

An additional benefit of using original source material to develop students' algebraic understanding is provided through the selection of excerpts which draw attention to mathematical subtleties which some modern texts may take for granted. In keeping with the historical record, this generally leads to a more concrete approach than is typically found in current texts. For example, to provide historical and mathematical context for the concept of an abstract group which was first defined in an 1854 paper by Arthur Cayley, students begin this project by studying specific mathematical systems (e.g., roots of unity, permutations groups) which were well-developed prior to Cayley's explicit recognition of their common structure. Cayley's own references to these and other specific nineteenth century appearances of the group concept in this paper further render it a powerful lens on the process and power of mathematical abstraction that more standard textbooks cannot provide.

A table of contents and brief description of the material developed in each section for the current edition of the project follows.

- Introduction
  This section includes a broad overview of the historical background of group theory in the theory of equations.

- Section 1: Roots of unity, permutations and equations: J. L. Lagrange
  The primary objective of this section is introduce students to two concrete examples of finite groups while providing some historical context for later portions of the paper. The notion of a 'resolvent equation' is introduced and developed in several exercises, but does not form focal point for later material. Instructors can therefore choose to safely omit several of these exercises as desired, or in the interest of time.

  - Subsection 1.1: Roots of unity in Lagrange's analysis
    Concepts related to finite cyclic groups are introduced in the context of Lagrange's writing on the solution of equations.
  - Subsection 1.2: Permutations of roots in Lagrange's analysis
    The concept of a permutation is introduced in the context of Lagrange's writing on the solution of equations.

- Section 2: An independent theory of permutations: A. Cauchy
  The primary objective of this section is to introduce aspects of the theory of finite groups within the concrete context of permutation groups.

- Subsection 2.1: Multiplication of permutations in Cauchy's theory
  Basic properties of permutation multiplication, including decomposition into disjoint cycles and even/odd permutations,
- Subsection 2.2: 'Systems of conjugate permutations' in Cauchy's theory
  The definition and elementary theory of permutation groups ('system of conjugate permutations' in Cauchy's terminology) are introduced in the context of Cauchy's writing on permutations. Cauchy's proof of Lagrange's Theorem for groups of permutations is studied in detail. The concrete language of this proof allows students to develop an understanding of its meaning without becoming lost in the abstraction of cosets, partitions and equivalence relations, while its complete generalizability of the proof to any finite group prepares them to make the transition to that level of abstraction later in the project/course. Modern notation for S_n and A_n are introduced.

- Section 3: The first paper on abstract group theory: A. Cayley
  The primary objective of this section is to extend the concrete examples of groups encountered in Sections 1 and 2 by studying the 1854 paper by Arthur Cayley in which a defintion of an abstract (finite) group appears for the first time: *On the theory of groups, as depending on the symbolic equation* $\theta^n = 1$

  - Subsection 3.1: Cayley's Definition of an Abstract Group
    The definitions of group and subgroup are developed, and basic group properties (e.g., abelian, cyclic, uniqueness of identity and inverses, cancellation) are introduced.
  - Subsection 3.2: Some Theorems on Groups
    Further topics in elementary group theory are developed, including basics results concerning the order of group elements, Lagrange's Theorem for finite groups and group isomorphisms.

- Section 4: Cayley's Classification of Groups of Small Order
  The concept of isomorphism is further explored in this section through a careful reading Cayley's classification of groups of order 4 and 6. The introduction to this section also discusses the classification problem for finite simple groups.

  - Subsection 4.1: Groups of Order 4
    Cayley's proof that there are only two groups of order 4 is studied in detail, and the Viergruppe is introduced. Cauchy's Theorem is also stated and used (but not proven).
  - Subsection 4.2: Groups of Order 6

    In addition to studying Cayley's proof that there are only two groups of order 6 is studied in detail, direct products and a classification of groups of order 8 is developed through exercises.

- Section 5: Concluding Remarks, and Cayley's Theorem Today
  A statement of the theorem now known as 'Cayley's Theorem' is stated, and its proof is sketched as an exercise.

- Appendix I: Optional exercises on de Moivre's and Euler's Formulas
  This optional appendix includes exercises related to the material on roots of unity from Section 1.

- Appendix II: Cosets, Lagrange's Theorem and Factor Groups
  This optional appendix includes an explicit examination of coset properties which are implicitly introduced in the main body of the project, and provides a definition of normal subgroup and factor group (without developing either concept in detail).