# Origins of Boolean Algebra in the Logic of Classes: George Boole, John Venn and C. S. Peirce

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## Notes to the instructor

This project is designed for an introductory or intermediate course in discrete or finite mathematics that includes a study of elementary set theory. Without explicitly introducing modern notation for operations on sets, the project develops a modern understanding of these operations and their basic properties within the context of early efforts to develop a symbolic algebra for logic. Although the project focuses on what would now be called 'introductory set theory,' it also lays the ground work for a more abstract discussion of boolean algebra as a discrete axiomatized structure. Accordingly, this project may also be used as an introduction to one or both of the companion projects described below in any course which considers boolean algebra from either a mathematical or computer science perspective.

Beginning with Boole's writings on the use of symbolic algebra to represent logical classes (Section 2), this project introduces the operations of logical addition (i.e., set union), logical multiplication (i.e., set intersection) and logical difference (i.e., set difference) and examines certain restrictions placed on their use by Boole which have since been lifted. The basic laws governing these operations are also introduced, as they were developed and justified by Boole; these justifications relied in part on his definitions of the operations and in part on the analogy of his symbols with those 'standard algebra.' The project then follows refinements to Boole's system made by Venn (Section 3) and Peirce (Section 4), with the level of abstraction steadily increasing through these sections. The project concludes (Section 5) with a summary of how Boole's 'Algebra of Logic' relates to elementary set theory as it is is typically presented today, and discusses how elementary set theory (when viewed as an algebraic structure) serves as a concrete example of a boolean algebra. Standard (undergraduate) notation and properties for set theory operations in use today are included and compared to standard (undergraduate) notation and axioms for a boolean algebra in that section. Issues related to language use and set operations which pose difficulties for many students, but which are ignored or unrecognized by contemporary textbook authors, are also explicitly considered in the writings of Boole and Venn (Sections 3 and 4), and further explored through the project question in those sections.

By following the refinements made to Boole's symbolic algebra in the hands of Venn and Peirce, this project provides students with an opportunity to witness first-hand *how* the process of developing and refining a mathematical system plays out, the ways in which mathematicians make and explain their choices along the way, and how standards of rigor in these regards have changed over time. Thus, in addition to developing the properties of set theory as a specific concrete example of a boolean algebra, the project is able to explore a variety of mathematical themes, including the notion of an inverse operation, the concept of duality, issues related to mathematical notation,

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and standards of rigor and proof. By following one or more of these themes through the project, instructors have considerable leeway in tailoring the project to their goals for a particular group of students.

Two companion projects on boolean algebra are also available as follow-up to this introductory project, either or both of which could also be used independently of the Boole-Venn-Peirce project. Additionally, either of the two companion projects could be used as a preliminary to or as a follow-up to the other companion project.

The companion project "Boolean Algebra as an Abstract Structure: Edward V. Huntington and Axiomatization" explores the early axiomatization of boolean algebra as an abstract structure through readings from Huntington's 1904 paper *Sets of Independent Postulates for the Algebra of Logic*. In addition to introducing the now standard axioms for the boolean algebra structure, the project illustrates how to use these postulates to prove some basic properties of boolean algebras. Specific project questions also provide students with practice in using symbolic notation, and encourage them to analyze the logical structure of quantified statements. The project also examines Huntington's use of the two-valued Boolean algebra on $K = \{0, 1\}$ — first studied by George Boole in his work on the logic of classes — as a model to establish the *independence* and *consistency* of one of his postulate sets. The final section of the project discusses modern (undergradute) notation and axioms for boolean algebras, and provides several practice exercises to reinforce the ideas developed in the earlier sections.

The companion project "Applying Boolean Algebra to Circuit Design: Claude Shannon," based on Shannon's ground-breaking paper *A Symbolic Analysis of Relay and Switching Circuits*, begins with a concise overview of the two major historical antecedents to Shannon's work: Boole's original work in logic and Huntington's work on axiomatization. The project then develops standard properties of a boolean algebra within the concrete context of circuits, and provides students with practice in using these properties to simplify boolean expressions. The two-valued boolean algebra on $K = \{0, 1\}$ again plays a central role in this work. The project closes with an exploration the concept of a 'disjunctive normal form' for boolean expressions, again within the context of circuits.

Implementation with students of any of these projects may be accomplished through individually assigned work, small group work and/or whole class discussion; a combination of these instructional strategies is recommended in order to take advantage of the variety of questions included in the project.