Discrete Mathematics Qual Exam (Spring 2018)
(This is a closed-book exam)

Question 1 (30 points)

(a) (4 points) Let p and q be boolean variables. Rewrite $\neg(p \to q)$ into an equivalent expression that uses operators $\neg$, $\wedge$ and $\vee$. (*Note: you are not required to use all the three operators. You may choose to use only a subset of the three operators.*)

Answer: $p \wedge \neg q$

(b) (3 points) Rewrite the statement $\neg\forall x\ p(x)$ into an equivalent statement using the existential quantifier.

Answer: $\exists x\ \neg p(x)$.

(c) (3 points) Rewrite the statement $\neg\exists x\ p(x)$ into an equivalent statement using the universal quantifier.

Answer: $\forall x\ \neg p(x)$.

(d) (4 points) Give the contrapositive version of the statement $p \to q$.

Answer: $\neg q \to \neg p$.

(e) (6 points) Are the two statements $(p \to q) \to r$ and $p \to (q \to r)$ equivalent? Give a proof for your answer.

Answer: No. We can prove the claim by constructing a truth table:

```
   ( (p ->  q)  ->  r ) <-> ( p -> (q  -> r) )
--------------------------------------------
      T  T  T   T   T    T    T  T  T  T  T
      T  T  T   F   F    T    T  F  T  F  F
      T  F  F   T   T    T    T  T  F  T  T
      T  F  F   T   F    T    T  T  F  T  F
      F  T  T   T   T    T    F  T  T  T  T
      F  T  T   F   F    F    F  T  T  F  F
      F  T  F   T   T    T    F  T  F  T  T
      F  T  F   F   F    F    F  T  F  T  F
```

(f) (10 points) Goldbach's conjecture states that every even integer greater than 2 can be written as the sum of two primes. In a logic statement, we can write it as

$$\forall n\ (((n > 2) \wedge (n \text{ is even})) \to \exists p\ \exists q\ (\text{prime}(p) \wedge \text{prime}(q) \wedge (n = p + q)))$$

Give the negation of the Goldbach's conjecture as a logical statement that makes use of the term $n \neq p + q$.

Answer: $\exists n\ (n > 2) \wedge (n \text{ is even}) \wedge \forall p\ \forall q\ (\text{prime}(p) \wedge \text{prime}(q) \to (n \neq p + q))$
or
$\exists n\ (n > 2) \wedge (n \text{ is even}) \wedge \forall p\ \forall q\ (\neg\text{prime}(p) \vee \neg\text{prime}(q) \vee (n \neq p + q))$

Question 2 (40 points)

Let Σ be a set of symbols. For example, $\Sigma = \{a, b\}$. We define the concept of strings over Σ as follows:

> Base case: ϵ is a string.
>
> Inductive case: If $x \in \Sigma$ and s is a string, then $x \cdot s$ is a string.
> (*A usual convention is to write $x \cdot s$ as $x\,s$ by omitting the $\cdot$ symbol.*)

(a) (15 points) Give an inductive definition of concatenation operation $\circ$ of two strings. Example: the concatenation of strings aba and $bbab$, denoted as $aba \circ bbab$, gives $ababbab$. (*Hint: define inductively on the first argument.*)

Answer: Base case: $\epsilon \circ t = t$. Inductive case: $(x \cdot s) \circ t = x \cdot (s \circ t)$.

(b) (25 points) Give an induction proof that string concatenation is associative: $(r \circ s) \circ t = r \circ (s \circ t)$. (*Hint: induction on r.*)

Answer: Base case: When $r = \epsilon$, $(\epsilon \circ s) \circ t = s \circ t = \epsilon \circ (s \circ t)$. Induction hypothesis: Assume that $(r \circ s) \circ t = r \circ (s \circ t)$. Induction step: $((x \cdot r) \circ s) \circ t = (x \cdot (r \circ s)) \circ t) = x \cdot ((r \circ s) \circ t)$ by the definition of concatenation. Next, by the induction hypothesis, the expression is rewritten to $x \cdot (r \circ (s \circ t))$, which equals to $(x \cdot r) \circ (s \circ t)$ by the definition of concatenation.

Question 3 (20 points)

Suppose we have two biased coins, named coins 1 and 2, where the probabilities of getting a head from a coin throw are p_1 and p_2 respectively. We start by throwing coin 1 until we get a head, then followed by throwing coin 2 until we get a head.

(a) (7 points) Let X_1 be the number of coin 1 throws for getting the first head. Let $\texttt{prob}(X_1 = n)$ denote the probability that n coin 1 throws are needed to get the first head. Express $E(X_1)$ using summation in terms of $\texttt{prob}(X_1 = n)$ and n. *Note: There is no need to express your answer in terms of p_1.*

Answer: $E(X_1) = \sum_{n=1}^{\infty} n \cdot \texttt{prob}(X_1 = n)$

(b) (7 points) What is the probability that the total number of coin throws (number of coin 1 throws and the number of coin 2 throws) is exactly 5? *Give an expression for your answer. You do not have to simplify the expression.*

Answer: $p_1 q_2^3 p_2 + q_1 p_1 q_2^2 p_2 + q_1^2 p_1 q_2 p_2 + q_1^3 p_1 p_2$ where $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$.

(c) (6 points) Let $X = X_1 + X_2$ where X_1 is defined previously, and X_2 is the number of coin 2 throws for getting the first head. Express $E(X)$ in terms of $E(X_1)$ and $E(X_2)$. (*No justification is needed.*)

Answer: $E(X) = E(X_1) + E(X_2)$.

Question 4 (10 points)

We want to calculate $\sum_{i=1}^{\infty} \sum_{j=1}^{i} \frac{1}{2^{i+j}}$.

(a) (5 points) You are asked to rewrite the expression by switching the order of the two summations. That is, express the expression in the following form by determining what values to pick for the question marks:

$$\sum_{j=?}^{?} \sum_{i=?}^{?} \frac{1}{2^{i+j}}$$

Answer: $\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \frac{1}{2^{i+j}}$

(b) (5 points) Compute the value of the expression that you give for part (a).
You are required to simplify the answer to a single number.

Answer:

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \frac{1}{2^{i+j}} &= \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \frac{1}{2^i} \frac{1}{2^j} \\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{i=j}^{\infty} \frac{1}{2^i} \\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \sum_{k=0}^{\infty} \frac{1}{2^{j+k}} \\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{2^j} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{2^j} \frac{1}{1-\frac{1}{2}} \\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{2^j} 2 \\
&= 2 \sum_{j=1}^{\infty} \left(\frac{1}{4}\right)^j \\
&= 2 \, \frac{1}{4} \, \frac{1}{1-\frac{1}{4}} \\
&= \frac{2}{3}
\end{aligned}$$