Discrete Mathematics Qual Exam (Spring 2016)
Closed book exam

Question 1 (30%)

Prove by mathematical induction on n that if $d_1 + d_2 + \ldots + d_n = 2(n-1)$ where $d_1, d_2, \ldots, d_n \geq 1$ and $n \geq 2$, then there exists a tree with n nodes whose degrees are exactly $d_1, d_2, \ldots, d_n$.

Hint: To argue that an undirected graph is a tree, you have to explain that the graph is connected and acyclic.

Answer:

Base case: when $n = 2$. Since $d_1 + d_2 = 2(2-1) = 2$ and $d_1, d_2 \geq 1$, we deduce that $d_1 = d_2 = 1$. There exists a tree with 2 nodes and 1 edge that satisfy the degree requirements.

Induction hypothesis: Assume that the statement is verified for any n positive integers which sum is $2(n-1)$, where $n \geq 2$.

Induction step: Consider any $n+1$ positive integers which sum is $2n$.

As $n \geq 2$, we have $2(n+1) > 2n > n+1$, which implies that there exists a vertex with degree larger than 1, and another vertex with degree 1.

Without loss of generality, we let $d_{n+1} = 1$ and $d_n > 1$. We define $d'_n = d_n - 1$. Then $d_1, d_2, \ldots, d_{n-1}, d'_n \geq 1$ and $d_1 + d_2 + \ldots, d_{n-1} + d'_n = 2n - 2 = 2(n-1)$. By the induction hypothesis, there exists a tree with degree sequence $d_1, d_2, \ldots, d_{n-1}, d'_n$. By extending the vertex with degree d'_n with one more edge to a new vertex, we still have a tree which degree sequence is now $d_1, d_2, \ldots, d_{n-1}, d_n, d_{n+1}$, where $d_{n+1} = 1$.

Question 2 (5% + 20%)

Suppose that we are given a key k to search for in a hash table with positions $0, 1, 2, \ldots, m-1$, and suppose that we have a hash function h mapping the keys into the set $\{0, 1, 2, \ldots, m-1\}$. The search scheme is as follows:

1. Compute the value $j = h(k)$, and set $i = 0$.
2. Probe in position j for the desired key k. If you find it, or if this position is empty, terminate the search.

3. Set $i = i+1$. If i now equals m, the table is full, so terminate the search. Otherwise, set $j = (i + j) \bmod m$, and return to step 2.

Assume that m is a power of 2.

(a) Express the value of j in step 2 in terms of h, k, i and m. (Note: if your answer involves a summation, be sure to rewrite it in a closed-form formula.)

Answer: $j = (h(k) + 1 + 2 + \ldots + i) \bmod m = \left(h(k) + \frac{i(i+1)}{2}\right) \bmod m$.

(b) Prove that this algorithm examines every table position in the worst case.

Hint: Show that the j values are distinct for $i = 0, 1, 2, \ldots, m-1$. Consider two j values that correspond to i and i' where $0 \le i < i' \le m-1$. Verify that the difference between the two j values is $((i'+i+1)(i'-i)/2) \bmod m$. Next, use case analysis to argue that $((i'+i+1)(i'-i)/2) \neq 0 \bmod m$.

Answer:

We want to argue that $i(i+1)/2 \bmod m \neq i'(i'+1)/2 \bmod m$ for $0 \le i < i' \le m-1$.
Equivalently, we want to show that $(i'(i'+1)/2 - i(i+1)/2) \bmod m \neq 0$,
which is the same as showing $((i'+i+1)(i'-i)/2) \bmod m \neq 0$.

- Case 1. $(i'+i+1)$ is odd, $(i'-i)$ is even.
  As $(i'+i+1) > 1$ is odd, and $0 < (i'-i)/2 < m$,
  $(i'+i+1)(i'-i)/2$ cannot be a multiple of m, which is a power of 2.
- Case 2. $(i'+i+1)$ is even, $(i'-i)$ is odd, $(i'-i) > 1$.
  As $0 < (i'+i+1)/2 < m$ and $(i'-i)$ is an odd number not equal to 1,
  $(i'+i+1)(i'-i)/2$ cannot be a multiple of m, which is a power of 2,
- Case 3. $(i'+i+1)$ is even, $(i'-i) = 1$.
  $(i'+i+1)(i'-i)/2 = i'$ cannot be a multiple of m, since $0 < i' \le m-1$.

Question 3. (4% + 4% + 4% + 4% + 7% + 7%)

Let X and Y be random variables over a **finite** sample space S.

(a) Define the concept of the expected value $E(X)$.

Answer:

$E(X) = \sum_{x \in S} x \cdot \text{Prob}\{X = x\}$.

(b) Define the concept that X and Y are independent.

Answer:

$\forall x, y \in S$, $\text{Prob}\{X = x \text{ and } Y = y\} = \text{Prob}\{X = x\} \cdot \text{Prob}\{Y = y\}$.

(c) Define $X + Y$. Let $Z = X + Y$. How is $\texttt{Prob}[Z = z]$ calculated?

Answer:

Let $Z = X + Y$. For $z \in S$, $\text{Prob}(Z = z) = \sum_{x,y \in S, z=x+y} \text{Prob}(X = x \text{ and } Y = y)$.

(d) Define XY. Let $Z = XY$. How is $\texttt{Prob}[Z = z]$ calculated?

Answer:

Let $Z = XY$. For $z \in S$, $\text{Prob}(Z = z) = \sum_{x,y \in S, z=xy} \text{Prob}(X = x \text{ and } Y = y)$.

(e)

Are there random variables X and Y that are not independent such that $E(X+Y) \neq E(X) + E(Y)$? If your answer is yes, then define such random variables X and Y, show that X and Y are not independent and that $E(X + Y) \neq E(X) + E(Y)$. If your answer is no, then give a proof that for any variables X and Y (that are not necessarily independent), $E(X + Y) = E(X) + E(Y)$.

Answer:

$$
\begin{aligned}
& E(Z) \\
= \; & E(X+Y) \\
= \; & \textstyle\sum_{z \in S} z \cdot \text{ Prob}(Z = z) \\
= \; & \textstyle\sum_{z \in S} z \cdot \sum_{x,y \in S, z=x+y} \text{ Prob}(X = x \text{ and } Y = y) \\
= \; & \textstyle\sum_{z \in S} \cdot \sum_{x,y \in S, z=x+y} (x+y) \cdot \text{ Prob}(X = x \text{ and } Y = y) \\
= \; & \textstyle\sum_{x,y \in S} (x+y) \cdot \text{ Prob}(X = x \text{ and } Y = y) \\
= \; & \textstyle\sum_{x,y \in S} x \cdot \text{ Prob}(X = x \text{ and } Y = y) + \sum_{x,y \in S} y \cdot \text{ Prob}(X = x \text{ and } Y = y) \\
= \; & \textstyle\sum_{x \in S} x \cdot \sum_{y \in S} \text{ Prob}(X = x \text{ and } Y = y) + \sum_{y \in S} y \cdot \sum_{x \in S} \text{ Prob}(X = x \text{ and } Y = y) \\
= \; & \textstyle\sum_{x \in S} x \cdot \text{Prob}(X = x) + \sum_{y \in S} y \cdot \text{Prob}(Y = y) \\
= \; & E(X) + E(Y)
\end{aligned}
$$

There is a shorter way to answer this question. See Section 18.3 of the MIT notes that we adopted for CS278.

(f)

Are there random variables X and Y that are not independent such that $E(XY) \neq E(X)E(Y)$? If your answer is yes, then define such random variables X and Y, show that X and Y are not independent and that $E(XY) \neq E(X)E(Y)$. If your answer is no, then give a proof that for any variables X and Y (that are not necessarily independent), $E(XY) = E(X)E(Y)$.

Answer:

Yes. Let $S = \{0, 1\}$. We define Prob($X = 0$ and $Y = 0$) $= 0.1$, Prob($X = 0$ and $Y = 1$) $=$ 0.2, Prob($X = 1$ and $Y = 0$) $= 0.3$ and Prob($X = 1$ and $Y = 1$) $= 0.4$. Then $E(X) = 0.7$ and $E(Y) = 0.6$. But, $E(XY) = 0.4 \neq 0.42 = E(X)E(Y)$.

Question 4 (15%)

A full binary tree is either a single-node tree, or consisting of a node with two subtrees that are full binary trees.

How many 11-node full binary trees are there?
You are required to make use of the recurrence principle in the calculations.

Answer:

Let $f(n)$ denote the number of full binary trees of n nodes.

$f(1) = 1.$
$f(2) = 0.$
$f(3) = f(1)f(1) = 1.$
$f(4) = f(1)f(2) + f(2)f(1) = 0.$
$f(5) = f(1)f(3) + f(2)f(2) + f(3)f(1) = 1 + 0 + 1 = 2.$
$f(6) = f(1)f(4) + f(2)f(3) + f(3)f(2) + f(4)f(1) = 0.$
$f(7) = f(1)f(5) + f(2)f(4) + f(3)f(3) + f(4)f(2) + f(5)f(1) = 2 + 0 + 1 + 0 + 2 = 5.$
$f(8) = 0.$
$f(9) = f(1)f(7) + f(2)f(6) + f(3)f(5) + f(4)f(4) + f(5)f(3) + f(6)f(2) + f(7)f(1)$
$\quad = 5 + 0 + 2 + 0 + 2 + 0 + 5 = 14.$
$f(10) = 0.$
$f(11) = 14 + 5 + 2 * 2 + 5 + 14 = 42.$