## December 2020 Discrete Mathematics Qualifying Exam

Closed book, closed notes, no calculators.

1) (20 points) Prove using mathematical induction that

$$\sum_{i=2}^{n}(5i-7) = (5n-4)(n-1)/2 \text{ for all integers } n \geq 2.$$

Note: You have to use mathematical induction; if you prove the statement using any other method, it will not count.

Solution:
Proof (by mathematical induction):
Base case:
The statement is true for n=2 because LHS = $10-7=3$ and RHS = $\frac{(10-4)(2-1)}{2} = 3$.
Inductive step:
Suppose that for some integer $k \geq 2$, $\sum_{i=2}^{k}(5i-7) = \frac{(5k-4)(k-1)}{2}$. This is the inductive hypothesis.
We must show that $\sum_{i=2}^{k+1}(5i-7) = \frac{(5(k+1)-4)((k+1)-1)}{2}$.
LHS = $\sum_{i=2}^{k+1}(5i-7)$
= $\left(\sum_{i=2}^{k}(5i-7)\right) + (5(k+1)-7)$ by separating the last term
= $\frac{(5k-4)(k-1)}{2} + (5(k+1)-7)$ by inductive hypothesis
= $\frac{5k^2-4k-5k+4}{2} + (5k-2) = \frac{5k^2-9k+4+10k-4}{2} = \frac{5k^2+k}{2}$ by algebra
RHS = $\frac{(5(k+1)-4)((k+1)-1)}{2} = \frac{(5k+1)k}{2} = \frac{5k^2+k}{2}$.
LHS = RHS. Therefore, the statement is true for all integers $n \geq 2$.

2) Write a negation for each of the following statements:
(a) (6 points) "The variable S is undeclared and the data are out of order"
(b) (6 points) "If Andy was with Bob on the 1st, then Andy is innocent."

Solution:
(a) The variable S is declared or the data is not out of order.
(b) Andy was with Bob on the 1$^{st}$ and Andy is not innocent.

3) (12 points) Determine whether the following argument is valid or invalid. Include a truth table and a few words explaining why the truth table shows validity or invalidity.

If Hugo is a physics major or a math major, then he needs to take calculus.
Hugo needs to take calculus or Hugo is a math major.
Therefore, Hugo is a physics major or Hugo is a math major.

Solution:

Let p="Hugo is a physics major", m="Hugo is a math major", c="Hugo needs to take calculus". Then, the argument can be written as follows.

$(p \vee m) \rightarrow c$
$c \vee m$
$\overline{\therefore p \vee m}$

Truth table:

| p | m | c | $(p \vee m) \rightarrow c$ | $c \vee m$ | $p \vee m$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | F | T | T |
| T | F | T | T | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | T | F | F | T | T |
| F | F | T | T | T | F |
| F | F | F | T | F | F |

The argument is invalid. When p=F, m=F, c=T, both of the hypotheses are true but the conclusion is false.

4) Let S = {a,b,c}
   (a) (7 points) What is the power set of S? (List all its elements.)
   (b) (7 points) What is S×S (the Cartesian product of S and S)? (List all its elements.)

   Solution:

   (a) {{}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c}}
   (b) {(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)}

5) Let *S* be the set of all strings of 0's and 1's. Function *g*: *S* $\rightarrow$ Z (where Z is the set of integers) is defined as follows: for each string *s* in *S*,

   *g*(*s*) = the number of 1's in *s* minus the number of 0's in *s.*

   (a) (6 points) What is *g*(101011)? What is *g*(00100)?
   (b) (6 points) Is *g* one-to-one? Prove or give a counterexample.
   (c) (6 points) Is *g* onto? Prove or give a counterexample.

   Solution:
   (a) *g*(101011)=2, *g*(00100)=-3
   (b) No, g is not one-to-one. For example, two different strings 11100 and 00111 map to the same number: g(11100) = g(00111) = 1.
   (c) Yes, g is onto. For any integer n in Z, there exists a string from S that maps into n. For instance, string consisting of n 1's will map into n.

6) A club has seven members. Three are to be chosen to go as a group to a national meeting.

(a) (4 points) How many distinct groups of three can be chosen?
(b) (4 points) If the club contains four men and three women, how many distinct groups of three contain two men and one woman?
(c) (4 points) If the club contains four men and three women, how many distinct groups of three contain at most two men?
(d) (4 points) If the club contains four men and three women, how many distinct groups of three contain at least one woman?
(e) (4 points) If two members of the club refuse to travel together as part of the group (but each is willing to go if the other does not), how many distinct groups of three can be chosen?
(f) (4 points) If two members of the club insists on either traveling together or not going at all, how many distinct groups of three can be chosen?

Solution:
(a) $\binom{7}{3}$
(b) $\binom{4}{2}\binom{3}{1}$
(c) Number of distinct groups with no restriction: $\binom{7}{3}$. Number of distinct groups with more than 2 men, that is, with 3 men: $\binom{4}{3}$. Therefore, the number of groups that contain at most 2 men is $\binom{7}{3} - \binom{4}{3}$.
(d) Number of distinct groups with no restriction: $\binom{7}{3}$. Number of distinct groups with no women: $\binom{4}{3}$. Therefore, the number of groups that contain at least one woman is $\binom{7}{3} - \binom{4}{3}$.
(e) One solution: Number of distinct groups of three that contain the 1st person and do not contain the 2nd person: $\binom{5}{2}$. Number of distinct groups of three that contain the 2nd person and do not contain the 1st person: $\binom{5}{2}$. Number of distinct groups of three that that contain neither of them: $\binom{5}{3}$. Total: $\binom{5}{2} + \binom{5}{2} + \binom{5}{3} = 10 + 10 + 10 = 30$.
Another solution: Number of distinct groups with no restriction: $\binom{7}{3}$. Number of distinct groups that contain both members: $\binom{5}{1}$. Therefore, number of distinct groups that do not contain both of them at once is $\binom{7}{3} - \binom{5}{1} = 35 - 5 = 30$.
(f) Number of distinct groups with both of them traveling: $\binom{5}{1}$. Number of distinct groups of three that contain neither of them: $\binom{5}{3}$. Total: $\binom{5}{1} + \binom{5}{3} = 5 + 10 = 15$.