December 2016 Discrete Mathematics Qualifying Exam Solution

Closed book closed notes

1. (10 points) Given the following information about a computer program, find the mistake in the program.

   a. There is a missing semicolon or there is a syntax error in the first five lines.

   b. If there is a syntax error in the first five lines, then there is an undeclared variable or a variable name is misspelled.

   c. There is not an undeclared variable.

   d. If there is a misspelled variable name then there is an undeclared variable.

   Solution: There is a missing semicolon.
   One possible explanation:
   1) There is no misspelled variable name (by (d) and (c) and Modus Tollens).
   2) There is not an undeclared variable and there is no misspelled variable name (by (c) and (1) and definition of conjunction)
   3) It is not the case that there is an undeclared variable or a variable name is misspelled (by (2) and De Morgan's laws)
   4) There is no syntax error in the first five lines (by (b) and (3) and Modus Tollens)
   5) There is a missing semicolon (by (a) and (4) and elimination).

2. (10 points) Imagine that num_orders and num_instock are particular values that might occur during execution of a computer program. Write negation for the following statement:
   (num_orders ≤ 10 and num_instock > 100) or (10 < num_orders ≤ 20 and num_instock > 200)

   Solution:
   (num_orders > 10 or num_instock ≤ 100) and (num_orders ≤ 10 or num_orders > 20 or num_instock ≤ 200)

3. (20 points) Prove that if n and m are odd integers, then the difference of their squares $(n^2 - m^2)$ equals $8k$ for some integer k.

   Proof:
   Suppose n and m are odd integers. By definition of odd, $n = 2s + 1$ and $m = 2t + 1$ for some integers s and t.
   Then, $n^2 - m^2 = 4s^2 + 4s + 1 - 4t^2 - 4t - 1 = 4(s^2 - t^2 + s - t) = 4(s - t)(s + t + 1)$.
   Let $r = s - t$. Then, r is an integer because it is a difference of two integers. By the quotient-remainder theorem, $r = 2q$ (even) or $r = 2q + 1$ (odd) for some integer q.
   Case 1: $r = 2q$. Then, $n^2 - m^2 = 4 \cdot 2q(s + t + 1) = 8q(s + t + 1)$. Let $k = q(s + t + 1)$. Note that k is an integer because sum of integers is an integer and product of integers is an integer. Therefore, $n^2 - m^2 = 8k$, where k is an integer (as was to be shown).
   Case 2: $r = 2q + 1$. Then, $s + t + 1 = s - t + 2t + 1 = r + 2t + 1 = 2q + 1 + 2t + 1 = 2(q + t + 1)$.
   Therefore, $n^2 - m^2 = 4r(s + t + 1) = 8r(q + t + 1)$. Let $k = r(q + t + 1)$. Note that k is an integer because sum of integers is an integer and product of integers is an integer. Therefore, $n^2 - m^2 = 8k$, where k is an integer (as was to be shown).

4. (20 points) Use mathematical induction to prove the following statement for every positive integer n:

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}$$

Proof by math induction:

Base case. If n=1, then LHS = 1·3 = 3 and RHS = $\frac{1(1+1)(2+7)}{6} = 3$.

Inductive step. Assume the statement is true for some k>0. That is,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}.$$

We need to show that it is true for k+1. That is,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+1+2) = \frac{(k+1)(k+1+1)(2(k+1)+7)}{6}.$$

LHS = $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + (k+1)(k+1+2)$

= $(1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2)) + (k+1)(k+1+2)$ by separating the last term

= $\frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$ by substitution from the inductive hypothesis

= $\frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6} = \frac{(k+1)(k(2k+7)+6(k+3))}{6} = \frac{(k+1)(2k^2+7k+6k+18)}{6} = \frac{(k+1)(2k^2+13k+18)}{6}$

RHS = $\frac{(k+1)(k+1+1)(2(k+1)+7)}{6} = \frac{(k+1)(k+2)(2k+9)}{6} = \frac{(k+1)(2k^2+4k+9k+18)}{6} = \frac{(k+1)(2k^2+13k+18)}{6}$ = LHS as was to be shown.

5. a) (10 points) Let X={a, b, c} and Y={1,2,3,4}. How many functions are there from X to Y? How many of these functions are one-to-one? How many of these functions are onto?
   b) (10 points) Let X be a set with n elements and Y be a set with m elements, where n and m are positive integers. How many functions are there from X to Y? How many of these functions are one-to-one?

Solution.

a) There are $4^3 = 64$ functions from X to Y. Of these functions, 4·3·2 = 24 are one-to-one and 0 functions are onto.

b) There are m^n functions from X to Y. If n>m then there are no one-to-one functions. If n≤m, then there are m·(m-1)·(m-2)·…·(m-n+1) one-to-one functions.

6. (20 points) A customer is ordering a new desktop computer system. The choices are 21-inch, 23-inch, or 24-inch monitor (optional); 1TB or 2TB hard drive; 6GB or 8GB of RAM; Intel, AMD, or NVIDIA video card; 1-, 2-, or 3-year warranty. Also, there are five accessories: keyboard, mouse, webcam, headset, and speakers. Any combination of accessories may be included in an order (from no accessories to all five accessories).

a) How many different orders are possible? (Note that orders may be with and without a monitor. In addition to the choices for the monitor (if any), hard drive, RAM, video card, and warranty, each order includes some subset of the five accessories.)

b) How many different orders are possible that have a 2 TB hard drive and two accessories?

c) How many different machines (without accessories) can be ordered with a 21-inch monitor and 8GB of RAM?

d) How many different machines (without accessories) can be ordered if the customer does not want a 3-year warranty?

e) How many different machines (without accessories) can be ordered that have either 2TB hard drive or 3-year warranty or both (2TB hard drive and 3-year warranty)?

Solution.

a) There are 4 choices for the monitor (21-inch, 23-inch, 24-inch, or no monitor). There are 2 choices for hard drive. There are 2 choices for RAM. There are 3 choices for video card. There are 3 choices for warranty. There are 2^5 choices for accessories combinations as there are 2^5 subsets of a 5-element set. Therefore, there could be $4 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2^5$ different orders.

b) Two accessories can be selected in $\binom{5}{2}$ ways. There is only 1 choice for hard drive. Therefore, there are $4 \cdot 1 \cdot 2 \cdot 3 \cdot 3 \cdot \binom{5}{2}$ orders like that.

c) There is 1 choice for monitor and 1 choice for RAM. Therefore, there are $1 \cdot 2 \cdot 1 \cdot 3 \cdot 3$ different machines.

d) If the customer does not want a 3-year warranty, then there are 2 choices for warranty: 1-year and 2-year. Therefore, there are $4 \cdot 2 \cdot 2 \cdot 3 \cdot 2$ different machines.

e) There are $4 \cdot 1 \cdot 2 \cdot 3 \cdot 3$ different machines with 2TB hard drive. There are $4 \cdot 2 \cdot 2 \cdot 3 \cdot 1$ different machines with 3-year warranty. There are $4 \cdot 1 \cdot 2 \cdot 3 \cdot 1$ different machines with 2TB hard drive and 3-year warranty. The number of machines with either 2TB hard drive or 3-year warranty or both is the following:
$4 \cdot 1 \cdot 2 \cdot 3 \cdot 3 + 4 \cdot 2 \cdot 2 \cdot 3 \cdot 1 - 4 \cdot 1 \cdot 2 \cdot 3 \cdot 1$.