Automata Qual Exam (Fall 2013)
Answer ALL questions (Closed Book Exam)

1. (10 points + 10 points)

One can define an unambiguous context-free grammar based on either the concepts of leftmost derivation sequence or parse tree.

A CFG G is unambiguous if for every string w generated by the grammar, there is a unique leftmost derivation sequence for w.

A CFG G is unambiguous if for every string w generated by the grammar, there is a unique parse tree for w.

We want to argue that the two definitions are equivalent. To achieve the goal, you are asked to prove the following results:

(a) Using mathematical induction, show that each parse tree gives rise to a unique leftmost derviation sequence.

> *Sketch of the answer: We need to define a more general concept of parse tree for which the root symbol of the tree can be any nonterminal. Then we perform an induction on the structure of the generalized parse trees.*

(b) Using mathematical induction, show that each leftmost derivation sequence gives rise to a unique parse tree.

> *Sketch of the answer: Induction on the length of the generalized derivation sequnce for which the start symbol can be any nonterminal of the grammar.*

For parts (a) and (b), when giving a mathematical induction proof, you need to make it very clear in what parameter that your induction is based on.

2. (25 points)

In this question, you can assume without proof that the halting problem (or, the acceptance problem) for Turing machines is undecidable.

Show that the following problem is undecidable: "Given a Turing machine M, is $L(M)$ Turing-decidable?"

Note that a Turing-decidable language is also called a recursive language.

Sketch of the answer: We modify the proof that $REGULAR_{\mathtt{TM}}$ is undecidable (Theorem 5.3 of the textbook). M_2 is designed to recognize the non-Turing-decidable language $A_{\mathtt{TM}}$ if M does not accept w, and to recognize the Turing-decidable language Σ^* if M accepts w. But unlike the proof of Theorem 5.3, M_2 cannot wait to finish processing its input x before running M on w. The running of M_2 on x and the running of M on w need to be done in a time-sharing manner. If either run accepts, then M_2 accepts.

3. (5 points + 10 points + 10 points + 10 points)

Let $L_{\mathtt{prime}} = \{ \ 0^p \mid p \text{ is prime } \}$. In this question, you can assume without proof that $L_{\mathtt{prime}}$ is not a regular language.

(a) Is $L_k = \{ \ 0^q \mid k \le q, \ q \text{ is not a prime number } \}$ a regular language? Give a proof for your answer.

Sketch of the answer: No. Suppose L_k is regular. Then there is a DFA M for the complement of L_k. Consider $L_{\mathtt{prime}}^{<k} = L_{\mathtt{prime}} \cap \{0^q \mid q < k\}$, which is regular as it is a finite language. Let M' be a DFA for $L_{\mathtt{prime}}^{<k}$. We can recognize $L_{\mathtt{prime}}$ as follows: If the input has k or more 0's, we consult M to decide, otherwise we consult M'.

(b) Is $L_{\mathtt{prime}}L_k$ a regular language? Give a proof for your answer.

Sketch of the answer: Yes. As $\{0^2, 0^3\} \subseteq L_{\mathtt{prime}}$ and $\{0^{2r} \mid k \le 2r\} \subseteq L_k$, we have $L_{\mathtt{prime}}L_k \supseteq \{0^2, 0^3\} \cdot \{0^{2r} \mid k \le 2r\} \supseteq \{0^q \mid q \ge k+3\}$, which is a co-finite language. Thus, $L_{\mathtt{prime}}L_k$ is also co-finite; hence it is regular.

(c) Is $L_{\mathtt{prime}}L_{\mathtt{prime}}$ a regular language? Give a proof for your answer.

Sketch of the answer: No. Consider $L_{\mathtt{prime}}L_{\mathtt{prime}} \cap \{ \ 0^k \mid k \text{ is an odd number } \}$, which is $0^2(L_{\mathtt{prime}} - 0^2) = \{0^{p+2} \mid p \ge 3\}$. As $\{ \ 0^k \mid k$ is an odd number

} is regular but $\{0^{p+2} \mid p \geq 3\}$ *is not regular, we conclude that* $L_{\texttt{prime}}L_{\texttt{prime}}$ *is not regular.*

(d) Goldbach's conjecture states that every even integer greater than 2 can be expressed as the sum of two primes. Suppose we assume that Goldbach's conjecture is true. Show that $L_{\texttt{prime}}L_{\texttt{prime}}L_{\texttt{prime}}$ is regular.

Sketch of the answer: From $\{0^2, 0^3\} \subseteq L_{\texttt{prime}}$ *and* $\{0^{2k} \mid k \geq 2\} \subseteq L_{\texttt{prime}}L_{\texttt{prime}}$*, we deduce that* $L_{\texttt{prime}}L_{\texttt{prime}}L_{\texttt{prime}}$ *is co-finite, hence a regular language.*

4. (20 points)

Given languages L_1 and L_2 over the alphabet Σ, we say that L_1 is approximately equal to L_2, written as $L_1 \approx L_2$ in notation, if the two languages differ only on a finite number of strings.

Is it decidable, for any regular languages L_1 and L_2, whether $L_1 \approx L_2$? Justify your answer.

Sketch of the answer: As regular languages are closed under complementation and union, the difference between the two languages $(L_1 - L_2) \cup (L_2 - L_1)$ *can be determined. Next, as it is decidable if a regular language is finite, we can determine if* $L_1 \approx L_2$*.*

Note: you can make use of any standard results and proofs from the textbook without proving it. But you still have to make clear and accurate arguments.